4. Other Discretization Methods

4.1. Discontinuous Galerkin (dG) Methods

= Nonconforming FEM

4.1.1. Derivation of dG Variational Scheme

- Let us again consider the **model problem**

(1)
$$\boxed{\begin{array}{l}\text{Given } f \in L_2(\Omega), \text{ find } u \in V = H^1(\Omega) \text{ such that} \\ \qquad -\Delta u = f \text{ in } \Omega \subset \mathbb{R}^d \ (\dagger \wedge \mathrm{Lip}, d = 2, 3) \\ \qquad\quad u = g \text{ on } \Gamma = \partial\Omega \end{array}}$$

- **Solvability:**

Due to Lax-Milgram, there exists a unique weak solution $u \in V_0 = \overset{\circ}{H}{}^1(\Omega) \subset V = H^1(\Omega)$:

$$(1)_{VF} \qquad \underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}_{a(u,v)} = \underbrace{\int_\Omega f \cdot v \, dx}_{(f,v)_{0,\Omega}} \quad \forall v \in \overset{\circ}{H}{}^1(\Omega)$$

$$a(u,v) = (f,v)_{0,\Omega} =: \langle F, v \rangle \quad \forall v \in V_0$$

- **Exercise 4.1:** (see also Chapter 2)

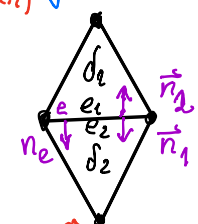
Show that $\nabla u \in H(\mathrm{div}, \Omega)$ and $\mathrm{div}\, \nabla u = -f \in L_2(\Omega)$!
Therefore, $u \in \overset{\circ}{H}{}^1(\Omega) \cap H^\Delta(\Omega)$, where
$H^\Delta(\Omega) := \{ u \in H^1(\Omega) : \Delta u = \mathrm{div}\, \nabla u \in L_2(\Omega) \}$.

"Continuity" properties in the corresponding trace spaces (see also Chapter 2):

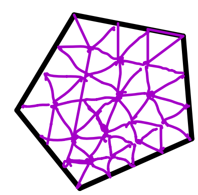
$u \in H^1(\Omega) \cap H^4(\Omega)$: $(1)_{VF}$ and $\frac{\partial u}{\partial n} := \nabla u \cdot n$

are "continuous" across interfaces in the following sense:

- $[u]_e := \underset{u|_{e_1}}{u_1} \vec{n}_1 + \underset{u|_{e_2}}{u_2} \vec{n}_2 = \underbrace{(u_1 - u_2)}_{[u]_e \in H^{1/2}(e)} \underset{n_e}{\vec{n}_e} \overset{u \in H^1(\Omega)}{=} 0$, $\quad \vec{n}_e = \vec{n}_1$
- $[\nabla u]_e := \nabla u_1 \cdot \vec{n}_1 + \nabla u_2 \cdot \vec{n}_2 = \underbrace{(\nabla u_1 - \nabla u_2) \cdot \vec{n}_e}_{\in H^{-1/2}} = 0$ $\quad \nabla u \in H(\mathrm{div}, \Omega)$



$u \in H^{s > 3/2}(\Omega)$
$u \in H^{s > 3/2}(\mathcal{T}_h) \xrightarrow[s > 3/2]{\text{Ch. 2}} \in L_2(e)$



dG-Notations:

Let $\mathcal{T}_h := \{\delta_r : r \in R_h\}$ be a regular triangulation of $\Omega \subset \mathbb{R}^{d=2}$:

For $s > 0$, we define the **broken Sobolev-spaces**:

$$H^s(\mathcal{T}_h) := \{ v \in L_2(\Omega) : v|_\delta \in H^s(\delta) \ \forall \delta \in \mathcal{T}_h \},$$

with $\|v\|^2_{H^s(\mathcal{T}_h)} := \sum_{\delta \in \mathcal{T}_h} \|v\|^2_{H^s(\delta)}$, $(u, v)_{H^s(\mathcal{T}_h)} = \sum_{\delta \in \mathcal{T}_h} (v, u)_{H^s(\delta)}$.

Furthermore, we define

- the jumps (differences)

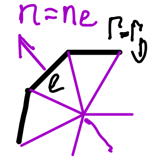
  $[v]_e := v_1 n_1 + v_2 n_2 = \underbrace{(v_1 - v_2)}_{[v]_e} n_e$ – vector, $e \in E_h$ ← Inner edges

  – scalar

- the mean values (avarages)

  $\{\nabla v\}_e := \frac{1}{2}(\nabla v_1 + \nabla v_2)$ – vector, $e \in E_h$

- $[v]_e := v \cdot n$, $\quad \{\nabla u\} := \nabla v$, $\quad e \in \partial E_h \setminus E_h$



dG - Variational identity ($\Rightarrow$ consistency):

- Let $u \in V_{g=0} \cap H^{\Delta}(\Omega)$ be the solution of $(1)_{VF}$, and let $u, v \in H^s(\mathcal{T}_h)$, $s > 3/2$, then

$f = -\operatorname{div} \nabla u$ in $L_2(\Omega)$

$$(2) \quad (f, v)_{0,\Omega} = \sum_{\delta \in \cancel{\mathcal{T}_h}} (f, v)_{\cancel{0,}\delta} =$$

integration by parts

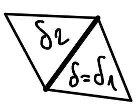
$$= \sum_{\delta} (\operatorname{div} \nabla u, v)_\delta =$$

$\in H^{-1/2}(\partial\delta)$ $\quad v \in H^{1/2}(\partial\delta)$

$$= \sum_{\delta} \left[(\nabla u, \nabla v)_\delta - \langle \nabla u \cdot \vec{n}, v \rangle_{H^{-1/2}(\partial\delta) \times H^{1/2}(\partial\delta)} \right]$$

$\Uparrow$ $L_2(\partial\delta)$ since $u \in H^s(\delta)$ with $s > 3/2$.

$$= \sum_{\delta} \left[(\nabla u, \nabla v)_\delta - (\nabla u \cdot \vec{n}, v)_{\partial\delta} \right]$$



$$= \sum_{\delta} (\nabla u, \nabla v)_\delta - \sum_{\delta} (\{\nabla u\}, v\, n)_{\partial\delta}$$

$u : (1)_{VF}$!

$n = n_1 = n_e$ $\quad \nabla u_1 \cdot n_1$ (cont. of the fluxes)

$e \in E_h$: $\{\nabla u\}_e \cdot n \overset{\downarrow}{=} \frac{1}{2}(\nabla u_1 \cdot n_1 + \nabla u_2 \cdot n_1) = \nabla u_1 \cdot n_1 = \nabla u \cdot n$

$e \in \partial\delta \cap \partial\Omega$: $\{\nabla v\}_e = \nabla v$

$$= \sum_{\delta \in \mathcal{T}_h} (\nabla u, \nabla v)_\delta - \sum_{e \in \overline{E}_h} (\{\nabla u\}, [v])_e$$

$v \cdot n$ if $e \in \partial\delta \cap \Gamma$

$$\sum_{\delta \in \mathcal{T}_h} (\{\nabla u\}, v \cdot n)_{\partial\delta} = \sum_{e \in \overline{E}_h} (\{\nabla u\}, [v])_e$$

$v_1 n_1 + v_2 n_2$ for $e \in E_h$

$\boxed{[u]_e = 0}$

$$= \sum_{\delta \in \mathcal{T}_h} (\nabla u, \nabla v)_\delta - \sum_{e \in \overline{E}_h} (\{\nabla u\}, [v])_e + \beta \sum_{e \in \overline{E}_h} ([u], \{\nabla v\})_e + \sum_{e \in \overline{E}_h} \frac{\alpha_e}{h_e} ([u]\,[v])_e$$

where $\beta = -1, 0, 1$, α_e - positiv parameters (L.4.5), $h_e = \int_e ds = |e|$.

- Define now the **dG bilinear forms**

(3) $$a_h(u,v) = a_{h,dG}(u,v) = a_{h,dG,\beta}(u,v)$$
$$:= \sum_{\delta\in\mathcal{T}_h} (\nabla u, \nabla v)_\delta - \sum_{e\in\bar{E}_h} (\{\nabla u\}, [v])_e$$
$$+\beta \sum_{e\in\bar{E}_h} ([u], \{\nabla v\})_e + \sum_{e\in\bar{E}_h} \frac{\alpha_e}{h_e}([u],[v])_e$$
$$\forall u, v \in H^s(\mathcal{T}_h), \; s > 3/2.$$

$\beta = -1$: $a_h(\cdot,\cdot)$ is symmetric!
$\beta = 1 \wedge \beta = 0$: $a_h(\cdot,\cdot)$ is non-symmetric!

- We can now formulate the following so-called **dG variational identity** (consistency)

(4) $$a_h(u,v) = (f, v)_{0,\Omega} \; \forall v \in H^s(\mathcal{T}_h)$$

that holds for the weak solution $u \in V_g$ of $(1)_{VF}$ provided that $u \in V_g \cap H^s(\mathcal{T}_h)$ for some $s > 3/2$.

- **Theorm 4.1:**
Let $s > 3/2$. Then the following two statements hold:
(a) Assume that the weak solution u of (1) belongs to $H^s(\mathcal{T}_h)$. Then u satisfies the dG VI (4).
(b) Conversly, if $u \in H^1(\Omega) \cap H^s(\mathcal{T}_h)$ satisfies (4), then u is also the solution of our VP $(1)_{VF}$.

Proof: (a) see derivation of (4) above (↑) ✓
(b) mms*

q.e.d.

dG-Scheme:

Let us define the dG space ($h \in \Theta$)

$$V_K(\mathcal{T}_h) = \{ v \in L_2(\Omega) : v|_\delta \in \mathbb{P}_K(\delta) \; \forall \delta \in \mathcal{T}_h \} \subset H^s(\mathcal{T}_h).$$

non-conform $\not\subset H^1(\Omega)$

Then the dG-Scheme reads as follows

$(4)_h$ $\exists!$?

$$\boxed{\begin{array}{l} \text{Find } u_h \in V_h := V_K(\mathcal{T}_h): \\ \quad a_h(u_h, v_h) = (f, v_h)_0 \quad \forall v_h \in V_h \end{array}}$$

Once the basis in $V_h = \operatorname{span}\{p^{(i)} : i \in \overline{\omega}_h\}$ is chosen, the dG scheme is equivalent to the linear system

$\underline{(4)}_h$

$$\boxed{\text{Find } \underline{u}_h \in \mathbb{R}^{N_h} : \; K_h \underline{u}_h = \underline{f}_h \text{ in } \mathbb{R}^{N_h}}$$

Remark 4.2:

1. The Dirchlet BC $u = 0$ is incorporated in (4) resp. $(4)_h$! This technique is connected with Nitsche! Show this statement! How would you incorporate an inhomogeneous Dirichlet BC $u = g$ on Γ?
2. $\beta = -1$: SIPG = Symmetric Interior Penalty Galerkin,
   $\beta = +1$: NIPG = Non-symmetric IPG ($\alpha_e = 0$!)
   $\beta = 0$: IIPG = Incomplete IPG

Consistency and Galerkin-Orthogonality:

Since $\tilde{V}_h = \tilde{V}_K(\tilde{\mathcal{T}}_h) \subset H^s(\mathcal{T}_h)$, Theorem 4.1 (a) yields consistency

(5) $$a_h(\overset{(1)_{VF}}{u}, v_h) = (f, v_h)_{0,\Omega} \ \forall v_h \in \tilde{V}_h \subset H^s(\tilde{\mathcal{T}}_h)$$

provided that the solution u of $(1)_{VF}$ belongs to $\tilde{V}_g \cap H^s(\mathcal{T}_h)$ for some $s > 3/2$.

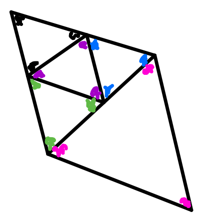
$(4)_h$ and (5) imply Galerkin-orthogonality

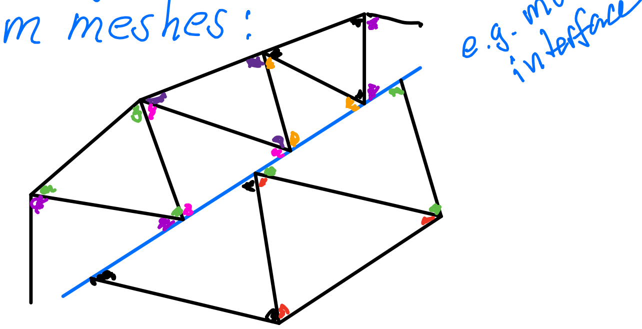
(6) $$a_h(u - u_h, v_h) = 0 \ \forall v_h \in \tilde{V}_h.$$

- **Remark 4.3:** Pros & Cons

1. **Pros:**

+ Variational handling of hanging nodes and non-conform meshes:



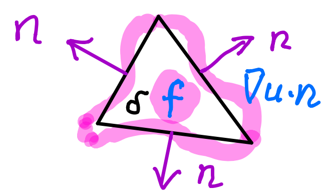


Adaptivity and treatment of moving interfaces (electric motors) is much easier!

+ block(δ_r) diagonal mass matrices:
  $\longrightarrow$ well suited for time-dependent problems!
+ natural upwinding for convection dominated problems!
+ conservative: test with $v_h(x) = \begin{cases} 1, & x \in \bar{\delta} \\ 0, & \text{otherwise} \end{cases} \in V_h$!

$$\Longrightarrow \int_{\delta} f \, dx = - \int_{\partial\delta} \nabla u \cdot n \, ds$$

dG-FEM $\longleftrightarrow$ FVM



+ ...

2. **Cons:**

- increasing number of global dofs!
  How to overcome this drawback?
  $\longrightarrow$ DD-dG (Nitsche)
  $\longrightarrow$ Hybriditation

δ $\qquad$ $\partial\delta$

- Larger stencils and non-locality due to coupling blocks!
- penalty parameters α_e, $e \in \bar{E}_h$!
- ...