3.6.2. The Residual Error Estimator

(Babuška & Rheinboldt, 1978)

- We now define,
  - for all elements $\delta_r \in \mathcal{T}_h$, the <u>residual of the PDE</u> ($\Delta u + f = 0$, or more general, $-Lu + f = 0$):

$$R_r = R_r(u_h) := \underbrace{\Delta u_h}_{=0} + f = -L u_h + f \quad \text{on } \delta_r, \; r \in R_h$$

$=0$ since u_h is linear on δ_r !

  - for all edges $e \in E_h := \bigcup_{r \in R_h} \{ e_1^{(r)}, e_2^{(r)}, e_3^{(r)} \} \setminus E(\Gamma_D)$

= set of all edges which do not belong to Γ_D = Γ for our example,

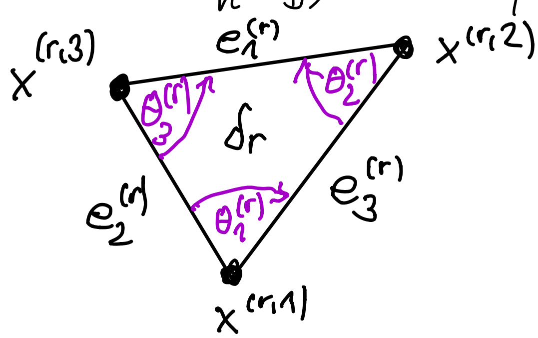
the <u>jumps of the fluxes</u> $\left(-\frac{\partial u}{\partial N}\right)$ across the edges

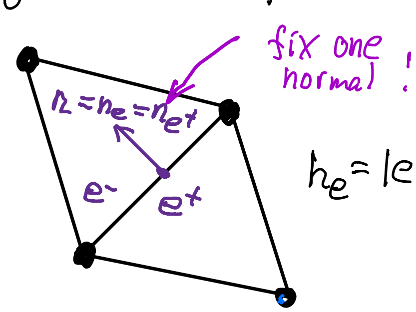
$$R_e = R_e(u_h) = \left[\frac{\partial u_h}{\partial n} \right]_e = [\nabla u_h]_e \cdot n_e$$

$$:= \left(\nabla u_h |_{e^+} - \nabla u_h |_{e^-} \right) \cdot n_e$$

$$= \nabla u_h |_{e^+} \cdot n_{e^+} + \nabla u_h |_{e^-} \cdot n_{e^-},$$

where $E_h(\Gamma_D) :=$ set of all edges on $\Gamma_D = \Gamma_1 = \Gamma$.





$h_e = |e|$

Theorem 3.19:

Ass.: Let $\mathcal{T}_h = \{\delta_r : r \in R_h\}$ be a shape regular triangulation of Ω, and let $u \in V_0$ and $u_h \in V_{0h}$ be the solution of (45) and $(45)_h$, respectively.

St.: Then the following a posteriori error estimates hold:

a) $$\| u - u_h \|_{1,\Omega} \leq (1 + c_F^2)^{1/2} | u - u_h |_{1,\Omega} \leq \bar{c} \Big\{ \sum_{r \in R_h} \eta_r^2(u_h) \Big\}^{1/2} = \bar{c}\, \eta(u_h),$$

b) $$\eta_r(u_h) \leq \underline{c} \Big\{ \| u - u_h \|^2_{1,U(\delta_r)} + \overbrace{\sum_{\delta_{r'} \subset U(\delta_r)} h_{r'}^2 \underbrace{\| f - f_h \|^2_{0,\delta_{r'}}}_{= O(h_{r'}^4)}}^{\leq h_r^2 \| f - f_h \|^2_{0,U(\delta_r)}} \Big\}^{1/2}$$

If $f|_{\delta_{r'}} \in H^2(\delta_r)$, then $= O(h_{r'}^4)$

HOT

where

$$\eta_r^2(u_h) = \eta_{\delta_r}^2(u_h) := h_r^2 \| \underbrace{R_r(u_h)}_{= \Delta u_h + f} \|^2_{0,\delta_r} + \frac{1}{2} \sum_{e \in \partial\delta_r \setminus \Gamma_D} h_e^1 \| \underbrace{R_e(u_h)}_{= \left[\frac{\partial u_h}{\partial n}\right]_e = [\nabla u_h]_e \cdot n_e} \|^2_{0,e}$$

$h_e = |e|$ = lengh of the edge e,

L_2 - projection

$f_h = P_h f \in V_{0h}$: $(f_h, v_h)_{0,\Omega} = (f, v_h)_{0,\Omega} \quad \forall v_h \in V_{0h}$,

$c = c(\Omega, c_\mathcal{T}) = c(\Omega, c_1, c_2, c_3) = \text{const} > 0$ - generic positive constant.

Proof: b) Efficiency: see Remark 3.21.1

a) We here only prove reliability!

From our general estimate (44) for Example (45), we have

Ex. (45): $\mu_1 = \mu_2 = 1$

$$\| u - u_h \|_{V_0} := | u - u_h |_{1,\Omega} \leq \frac{1}{\mu_1} \| F - A u_h \|_{V_0^* = H^{-1}(\Omega)} =$$

$$= \sup_{v \in \overset{\circ}{H}{}^1(\Omega) \setminus \{0\}} \frac{1}{|v|_{1,\Omega}} \sum_r \Big[\int_{\delta_r} (f + \Delta u_h)(v - v_h)\,dx - \int_{\partial\delta_r} \frac{\partial u_h}{\partial n_r} (v - v_h)\,ds \Big],$$

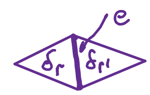
$\forall v \in \overset{\circ}{H}{}^1(\Omega)$

i.e. we have to estimate the numerator:

$$\sum_r \Big[\int_{\delta_r} (f + \Delta u_h)(v - v_h)\,dx - \int_{\partial\delta_r} \frac{\partial u_h}{\partial n_r} (v - v_h)\,ds \Big] \leq \bar{c}\, \eta_r(u_h) |v|_{1,\Omega}$$

Let $v \in V_0 = \mathring{H}^1(\Omega)$, and $v_h = I_h v \in V_{0h}$ its CLÉMENT- or SCOOT-ZHANG-quasiinterpolation (see Remark 3.20 and Ex. 3.14 for the necessary modification in $V_0 = \mathring{H}^1(\Omega)$!):

$$\Big| \sum_{r \in R_h} \int_{\delta_r} R_r(u_h)(v-v_h)\,dx - \sum_{e \in E_h} \int_e \Big[\frac{\partial u_h}{\partial n_e}\Big] \underbrace{(v-v_h)}_{=0 \text{ on } \Gamma_0 = \Gamma = \partial\Omega} ds \Big|$$



$(\nabla u_h|_{e+} - \nabla u_h|_{e-}) \cdot n_{e+}$

$$= \Big| \sum_{r \in R_h} \int_{\delta_r} R_r(u_h)(v-v_h)\,dx - \sum_{e \in E_h} \int_e \Big[\frac{\partial u_h}{\partial n_e}\Big]_e (v-v_h)\,ds \Big|$$

Δ-ineq. $\int$-Cauchy

$$\le \sum_{r \in R_h} \|R_r(u_h)\|_{0,\delta_r} \|v - v_h\|_{0,\delta_r} + \sum_{e \in E_h} \|R_e(u_h)\|_{0,e} \|v - v_h\|_{0,e}$$

Lemma 3.18

$$\le c \sum_{r \in R_h} \|R_r(u_h)\|_{0,\delta_r} h_r |v|_{1,U(\delta_r)} + c \sum_{e \in E_h} \|R_e(u_h)\|_{0,e} h_e^{1/2} |v|_{1,U(e)}$$

$$\le c \Big\{ \sum_{r \in R_h} h_r^2 \|R_r\|^2_{0,\delta_r} \Big\}^{1/2} \underbrace{\Big\{ \sum_{r \in R_h} |v|^2_{1,U(\delta_r)} \Big\}^{1/2}}_{\le c|v|_{1,\Omega}} + c \Big\{ \sum_{e \in E_h} h_e \|R_e\|^2_{0,e} \Big\}^{1/2} \underbrace{\Big\{ \sum_{e \in E_h} |v|^2_{1,U(e)} \Big\}^{1/2}}_{\le c|v|_{1,\Omega}}$$

T_h - shape regular:

$$\le c \Big[\Big\{ \sum_{r \in R_h} h_r^2 \|R_r\|^2_{0,\delta_r} \Big\}^{1/2} + \Big\{ \sum_{e \in E_h} h_e \|R_e\|^2_{0,e} \Big\}^{1/2} \Big] |v|_{1,\Omega}$$

$a + b = 1 \cdot a + 1 \cdot b \le \sqrt{2}\sqrt{a^2+b^2}$

$$\le c \Big[\sum_{r \in R_h} h_r^2 \|R_r\|^2_{0,\delta_r} + \sum_{e \in E_h} h_e \|R_e\|^2_{0,e} \Big]^{1/2} |v|_{1,\Omega}$$

$$= c \Big[\sum_{r \in R_h} \underbrace{\Big\{ h_r^2 \|R_r\|^2_{0,\delta_r} + \frac{1}{2} \sum_{e \in \partial\delta_r \setminus \Gamma_0} h_e \|R_e\|^2_{0,e} \Big\}}_{=: \eta_r^2(u_h)} \Big]^{1/2} |v|_{1,\Omega}$$

$$= c \Big[\sum_{r \in R_h} \eta_r^2(u_h) \Big]^{1/2} |v|_{1,\Omega} = c\, \eta(u_h)\, |v|_{1,\Omega}.$$

Therefore, we have proved the estimate

$$|u-u_h|_{1,\Omega} \underset{\mu_1=\mu_2=1}{\overset{\leq}{=}} \frac{1}{\mu_1}\|F-Au_h\|_{H^{-1}(\Omega)} \leq \frac{c}{\mu_1}\eta(u_h).$$

Due to Friedrichs' inequality, we also have

$$\|u-u_h\|_{1,\Omega} \leq \bar{c}\,\eta(u_h).$$

q.e.d.

- **Remark 3.20:**

The standard CLÉMENT-interpolant of a function $v \in \overset{\circ}{H}{}^1(\Omega)$ is not contained in V_{0h}, i.e.

$$I_h v = \sum_{i\in\bar{\omega}_h} (P_j v)\, p^{(j)}(x) \notin V_{0h},$$

but $(\bar{\omega}_h \mapsto \omega_h)$

$$\sum_{i\in\omega_h} (P_j v)\, p^{(j)}(x) \in V_{0h} \;!$$

Instead of the CLÉMENT-interpolant, one can also use the SCOTT-ZHANG-interpolant [Mathematics of Computation, 54 (190), 483-493, 1990)

- $\boxed{\textbf{Ex. 3.14*}}$ Let $v \in V_0 = \overset{\circ}{H}{}^1(\Omega)$ and $v_h = \sum_{i\in\omega_h} (P_j v)\, p^{(j)} \in V_{0h}$.

Show the interpolation error estimates

a) $\|v-v_h\|_{0,\delta_r} \leq c\, h_r\, |v|_{1,\omega(\delta_r)} \quad \forall r \in R_h$

b) $\|v-v_h\|_{0,e} \leq c\, h_e^{1/2}\, |v|_{1,\omega(e)} \quad \forall e \in E_h$

with some neric constant c !

Remark 3.21:

1. We refer to the literature, e.g. Braess (2013), p. 172f, for the proof of the efficiency estimate b)!
2. Other a posteriori error estimators:
   1) Estimators based on the solution of local Dirichlet problems:
      → Babuška-Rheinboldt (1978)
   2) Estimators based on the solution of local Neumann problems:
      → Bank-Weiser (1985)
   3) Estimators based on averaging ∇u_h:
      → Zienkiewic-Zhou (ZZ) indicator (1987)
   4) Hierarchical estimators:
      → Deuflhard-Leinen-Yserentant (1990)
   5) Functional type a posteriori error estimators:
      → Repin (1997)

Friedrichs

$$|u-v|_{1,\Omega} \lesssim \overline{M}(v,\sigma) := \|\sigma - \nabla v\|_{0,\Omega} + C_F(\Omega)\,\|f + \operatorname{div}\sigma\|_{0,\Omega}$$

$$\forall v \in \overline{V}_0 \quad \text{and} \quad \forall \sigma \in H(\operatorname{div},\Omega)$$

e.g. $\overset{\shortparallel}{u_h} \in \overline{V}_{0h} \subset \overline{V}_0$ $\qquad \overset{\shortparallel}{R_h}(\nabla u_h)$?

   6) Equilibrated error estimators
      → Braess-Schöberl (2008)
3. Standard literature on a priori error estimators:

[1] Verfürth R.: A review of a posteriori error estimation and adaptive mesh refinement techniques. Wiley-Teubner, 1996

[2] Repin S.: A priori error estimates for PDEs. RSCAM, de Gruyter, Berlin, 2008.

[...] ...

Repin's functional a posteriori error estimates:

Consider again the Dirichlet problem for the Poisson equation as model problem:

(45) Find $u \in V_0 = \overset{\circ}{H}{}^1(\Omega)$: $\int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx \quad \forall w \in V_0$.

Let $v \in V_g = V_0$ be an arbitrary "approximation" to u, e.g. the f.e. approximation u_h:

$$\int_\Omega \nabla(u-v)\cdot\nabla w \, dx = \int_\Omega (fw - \nabla v \cdot \nabla w)\, dx \quad \forall w \in V_0$$

Let $\sigma \in H(\mathrm{div},\sigma)$ be an arbitrary $H(\mathrm{div})$ function, i.e.

$$\int_\Omega (\mathrm{div}\,\sigma \cdot w + \sigma \cdot \nabla w)\, dx = 0 \quad \forall w \in V_0$$

Note: $\nabla u_h \notin H(\mathrm{div},\Omega)$!!!

$$\int_\Omega \nabla(u-v)\cdot\nabla w = \int_\Omega [(\mathrm{div}\,\sigma + f) w + (\sigma - \nabla v)\cdot \nabla w]\, dx$$

Set $w = u - v$: $\quad (\|\cdot\| = \|\cdot\|_{0,\Omega} = \|\cdot\|_{L_2(\Omega)})$

$$|u-v|_{1,\Omega}^{\cancel{2}} = \|\nabla u - \nabla v\|^{\cancel{2}} \overset{\text{Cauchy}}{\leq} \|\mathrm{div}\,\sigma + f\| \, \|u-v\| + \|\sigma - \nabla v\| \, \|\nabla(u-v)\|$$

Friedrichs

$$\leq \|\sigma - \nabla v\| \, \|\nabla\cancel{(u-v)}\| + c_F(\Omega)\, \|\mathrm{div}\,\sigma + f\| \, \|\nabla\cancel{(u-v)}\|$$

$$\Longrightarrow \|\nabla(u-v)\| \leq \|\sigma - \nabla v\| + c_F(\Omega)\, \|\mathrm{div}\,\sigma + f\|$$

$$=: \overline{M}_1(v,\sigma) \quad = \text{1st majorant}$$

$0 \le \left(\sqrt{\beta}\, a - \frac{1}{\sqrt{\beta}} b\right)^2 = \beta a^2 - 2ab + \frac{1}{\beta} b^2$

$(a+b)^2 \le (1+\beta)\, a^2 + \left(1+\frac{1}{\beta}\right) b^2 \quad \forall a,b \ge 0 \ \forall \beta > 0$

$$\Longrightarrow \|\nabla(u-v)\|^2 \le (1+\beta)\, \|\sigma - \nabla u\|^2 + \left(1+\frac{1}{\beta}\right) c_F^2 \, \|\operatorname{div}\sigma + f\|^2$$

$$=: \overline{M}_2(v, \sigma; \beta) \quad = \text{2nd majorant}$$

$$\Longrightarrow |u - \underset{u_h}{\underset{\shortparallel}{v}}|_{1,\Omega}^2 = \|\nabla(u-u_h)\|_{L_2(\Omega)} \le \inf_{\substack{\sigma \in H(\operatorname{div},\Omega) \\ \beta > 0}} \overline{M}_2(\overset{u_h}{\overset{\shortparallel}{v}}, \sigma, \beta)$$

$$\le \inf_{\substack{\sigma_h \in RT_h(\Omega) \subset H(\operatorname{div},\Omega) \\ \beta > 0}} \overline{M}_2(u_h, \sigma_h, \beta)$$

altern. min

Raviart-Thomas-f.e. space

OR $\sigma = \sigma_h = RT_h(\nabla u_h) \in H(\operatorname{div},\Omega)$

reconstruction

Goal-orientated a-posteriori error estimators:

In practice, instead of the solution $u \in V_g$ (or $u \in V_0$ after homogenization), we are often interested in the value $\ell(u) \in \mathbb{R}$ of a linear functional $\ell \in V_0^*$ on the solution $u \in V_0$, i.e. we need an a-posteriori error estimator of the error:

$$(46) \qquad |\ell(u) - \ell(u_h)| \leq \; ?$$

Examples:

1) $\ell(v) := \dfrac{(u-u_h, v)_0}{\|u-u_h\|_0}$: $\ell(u)-\ell(u_h) = \|u-u_h\|_0$

2) $\ell(v) := \int_{\Omega_0} v(x)\,dx$: average of v over $\Omega_0 \subset \Omega$

3) $\ell(v) := \int_{\Omega} \delta_\varepsilon(x-y)\, v(y)\,dx$: $\ell(u)-\ell(u_h) \approx u(y)-u_h(y)$

4) $\ell(v) := \int_{\Gamma_0} \frac{\partial v}{\partial N}\,ds$: Flux of v through $\Gamma_0 \subset \Gamma = \partial\Omega$?

etc.

To derive a-posteriori error estimates for (46), similar to the apriori L_2-estimate, we use a duality argument, i.e. we consider the auxiliary adjoint variational problem:

$$(47) \quad \text{Find } w \in V_0 : \; a(v, w) = \ell(v) \quad \forall v \in V_0$$

Now, take $v = u - u_h \in V_0$:

$$\ell(u) - \ell(u_h) = \ell(u-u_h) = a(u-u_h, w)$$

$$(\uparrow) \qquad = a(u-u_h, w-w_h) \quad \forall w_h \in V_{0h}$$

$$= \langle F, w-w_h \rangle - a(u_h, w-w_h)$$

$$\uparrow \quad \uparrow \qquad\qquad \uparrow \quad \uparrow$$

$$|\ell(u)-\ell(u_h)| = |\langle F, w-w_h\rangle - a(u_h, w-w_h)|$$

$$\overset{(1)}{\le} \sum_{r\in R_h} \|R_r(u_h)\|_{0,\delta_r}\|w-w_h\|_{0,\delta_r} + \sum_{e\in E_h}\|R_e(u_h)\|_{0,e}\|w-w_h\|_{0,e} \qquad (48)$$

$$= \sum_{r\in R_h} \underbrace{\left[\|R_r(u_h)\|_{0,\delta_r}\|w-w_h\|_{0,\delta_r} + \frac{1}{2}\sum_{e\in\partial\delta_r\setminus\Gamma_D}\|R_e(u_h)\|_{0,e}\|w-w_h\|_{0,e}\right]}_{=:\eta_{\delta_r}(u_h)}$$

$$= \sum_{r\in R_h}\eta_{\delta_r}(u_h) = \eta(u_h),$$ no constants, but w, w_h! dual weights

PDE: $Lu = f$ in Ω $\qquad -\Delta u = f$ in Ω Example

BC: $\ell u = g$ on Γ $\qquad u = 0$ on Γ

$R_r(u_h) = f - Lu_h = f + \Delta u_h$ PDE residual

$R_e(u_h) = [\frac{\partial u}{\partial N}]_e = [\frac{\partial u_h}{\partial n}]_e$ jumps in the fluxes

<u>Question:</u> How to determine the dual weights $w\in\tilde{V}_0$ and its FE approximation $w_h\in\tilde{V}_{0h}$?

① We proceed as in the proof of Theorem 2.19, i.e.

1) $w_h = \text{CLÉMENT}(w)$: $\|w-w_h\|_{0,\delta_r}\le c h_r |w|_{1,\omega(\delta_r)}$ $\ldots$

2) Find FE-solution $\tilde{w}_h\in\tilde{V}_{0h}$: (47) $\leadsto$ $|\tilde{w}_h|_{1,\omega(\delta_r)}\approx|w|_{1,\omega(\delta_r)}$

② We solve the "dual" problem (47) twice:

$$\left.\begin{array}{l} w_h\in\tilde{V}_{0h}: (47)_h \\ \tilde{w}_h\in V_{0h}^{HO}: (47)_h \quad \approx w\end{array}\right\} \|w-w_h\|_\bullet \approx \|\tilde{w}_h - w_h\|_\bullet$$

③ $$\left.\begin{array}{l}\text{Solve } (47)_h \leadsto w_h\in\tilde{V}_{0h} \\ w \approx \text{Higher-order reconstruction}(w_h) = \tilde{w}_h\end{array}\right\} -\text{"}-$$

Adaptive Mesh Refinement (~ Coarsening)

on the basis of the local error estimator

$$\eta_r(u_h) := \left[h_r^2 \, \| R_h(u_h) \|^2_{L_2(\delta_r)} + \frac{1}{2} \sum_{e \in \partial \delta_r \setminus \Gamma_D} h_e \, \| R_e(u_h) \|^2_{L_2(e)} \right]$$

$r \in \mathbb{R}_h$

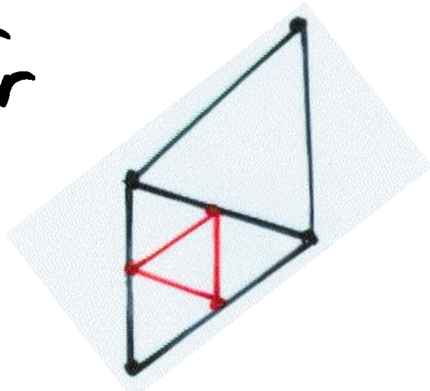
The principle of the uniform distribution of these local errors leads to the following ALGORITHM: AFEM

- Mark all δ_r : $\eta_r(u_h) \geq \Theta \max_{q \in \mathbb{R}_h} \eta_q(u_h)$

  e.g. with $\Theta = 0.7$

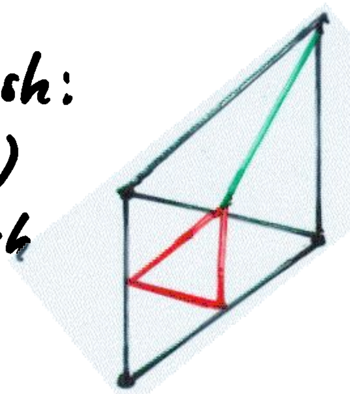
- Refine all marked δ_r

  red refinement



- Ensure the conformity of the mesh

  Rule ensuring the shape regularity of the mesh:
  "Do halven (bisect) an angle of the mesh only once!"



- New FE-calculation with the new mesh

AFEM solve → estimate → mark → refine / coarse