3.6. A posteriori Error Estimates and Adaptive Finite Element Methods (AFEM)

A trivial a posteriori error estimate:

Consider the variational problem ($g=0$, at least, after homog.)

$\exists!$ (1) Find $u \in \bar{V}_0$: $\underbrace{a(u,v)}_{\langle Au,v\rangle} = \langle F, v\rangle \ \forall v \in \bar{V}_0$: $Au = F$ in $\bar{V}_0^*$

$\boxed{\begin{array}{l} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{array}}$ Poisson

Example: $u \in \overset{\circ}{H}{}^1(\Omega)$: $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \ \forall v \in \overset{\circ}{H}{}^1(\Omega) = \bar{V}_0$ $(1)_{Ex}$

and its finite element discretization

$\exists!$ $(1)_h$ Find $u_h \in \bar{V}_{0h}$: $a(u_h, v_h) = \langle F, v_h\rangle \ \forall v_h \in \bar{V}_{0h}$

under the standard assumptions (33), $\bar{V}_h \subset \bar{V}$, $\bar{V}_{0h} \subset \bar{V}_0$, $\bar{V}_{gh} \subset \bar{V}_g$.

Then the following a posteriori error estimate is valid:

$$(44) \quad \boxed{\frac{1}{\mu_2} \| F - Au_h \|_{\bar{V}_0^*} \le \| u - u_h \|_{\bar{V}_0} \le \frac{1}{\mu_1} \| F - Au_h \|_{\bar{V}_0^*}}$$

Example: a) $H^{-1}(\Omega)$ $H^1(\Omega)$ b) $H^{-1}(\Omega)$

where $\| F - Au_h \|_{\bar{V}_0^*} := \sup_{v \in \bar{V}_0} \frac{\langle F - Au_h, v\rangle}{\|v\|_{\bar{V}_0}}$

$$= \sup_{v \in \bar{V}_0} \frac{\langle F, v\rangle - a(u_h, v)}{\|v\|_{\bar{V}_0}} = \sup_{v \in \bar{V}_0} \frac{a(u - u_h, v)}{\|v\|_{\bar{V}_0}}$$

Proof: $\|\cdot\| = \|\cdot\|_{V_0} \cong \|\cdot\|_V$ $\quad \|\cdot\|_{V_0} := |\cdot|_{1,\Omega} = |\cdot|_1$

b) $$\mu_1 \| u - u_h \|_{\bar{V}_0} \le \frac{a(u - u_h, u - u_h)}{\| u - u_h \|_{\bar{V}_0}} \le \sup_{v \in \bar{V}_0} \frac{a(u - u_h, v)}{\|v\|_{\bar{V}_0}} = \|F - Au_h\|_{\bar{V}_0^*}$$

a) $$\|F - Au_h\|_{\bar{V}_0} = \sup_{v \in \bar{V}_0} \frac{a(u - u_h, v)}{\|v\|_{\bar{V}_0}} \le \mu_2 \sup_{v \in \bar{V}_0} \frac{\|u - u_h\| \, \|v\|}{\|v\|} = \mu_2 \|u - u_h\|_{\bar{V}_0}$$

q.e.d.

Question: How to compute / estimate $\| F - Au_h \|_{\bar{V}_0^* = H^{-1}(\Omega)}$?

Idea: Road map for deriving residual a posteriori estimates:

$$\|F - Au_h\|_{\mathring{V}_0^*} = \sup_{v\in\mathring{V}_0} \frac{a(u-u_h,v)}{\|v\|} = \sup_{v\in\mathring{V}_0} \frac{a(u-u_h,v-v_h)}{\|v\|}$$

GO: $a(u-u_h,v_h)=0 \ \forall v_h \in \mathring{V}_{0h}$

$$= \sup_{v\in\mathring{V}_0} \frac{\langle F, v-v_h\rangle - a(u_h,v-v_h)}{\|v\|}$$

$\mathring{V}_0 = \mathring{H}^1(\Omega)$
$\|\cdot\|_{V_0} := |\cdot|_1 = |\cdot|_{H^1(\Omega)}$ is norm in $\mathring{V}_0$

$$\underset{(1)_{Ex}}{=} \sup_{v\in\mathring{V}_0=\mathring{H}^1(\Omega)} \frac{\int_\Omega f(v-v_h)\,dx - \int_\Omega \nabla u_h\cdot\nabla(v-v_h)\,dx}{|v|_1}$$

partial integration

$$= \sup_{v\in\mathring{H}^1(\Omega)} \frac{\sum_{r\in R_h}' \left[\int_{\delta_r} f(v-v_h)\,dx - \int_{\delta_r}\nabla u_h\cdot\nabla(v-v_h)\,dx\right]}{|v|_1}$$

$$= \sup_{v\in\mathring{H}^1(\Omega)} \frac{\sum_{r\in R_h}' \left[\int_{\delta_r} f(v-v_h)\,dx + \int_{\delta_r}\Delta u_h(v-v_h)\,dx - \int_{\partial\delta_r}\frac{\partial u_h}{\partial n}(v-v_h)\,ds\right]}{|v|_1}$$

$$= \sup_{v\in\mathring{H}^1(\Omega)} \frac{\sum_{r\in R_h}' \left[\int_{\delta_r} (f+\Delta u_h)(v-v_h)\,dx - \int_{\partial\delta_r}\frac{\partial u_h}{\partial n}(v-v_h)\,ds\right]}{|v|_1}$$

$(\downarrow)$: $d=1$: (mms) $\Omega=(a,b)$ $v\in\mathring{H}^1(\Omega) \hookrightarrow C^1(\bar\Omega)$, $v_h = Int_{\bar V_h}(u) \in \bar V_{0h}$ $\quad \forall v_h \in \mathring{V}_{0h}$

$$\le \sup_{v\in\mathring{H}^1(\Omega)} \frac{\sqrt{\sum_{r\in R_h}' \eta^2_{\delta_r}(u_h)}\ |v|_1}{|v|_1} = \sqrt{\sum_{r\in R_h}' \eta^2_{\delta_r}(u)} =: \eta_{\delta_r}(u_h)$$

$v_h \in \mathring{V}_{0h}$?

error estimator, where $\eta_{\delta_r}(u_h)$ is the error contribution

$v_h = Int_{\bar V_h}(v)$ is only possible in 1d since $\mathring{H}^1(\Omega) \not\subset C^1(\bar\Omega)$ for $d\ge 2$!

$v_h = I_h v$ - CLÉMENT or SCOTT-ZHANG - Interp. form the f.e. δ_r !

- For the sake of simplicity, we consider the **homogeneous Dirichlet problem for the Poisson equation** in a polygonal, bounded domain $\Omega \subset \mathbb{R}^2$:

(45)$_{CF}$ $\qquad -\Delta u = f$ in Ω and $u = 0$ on $\Gamma = \partial\Omega$.

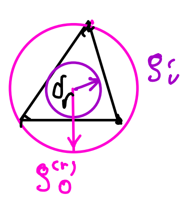
The corresponding variational formulation

(45)$_{VF}$ Find $u \in V_0 = \overset{\circ}{H}{}^1(\Omega)$: $\underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}_{\overset{\shortparallel}{a(u,v)}} = \underbrace{\int_\Omega f v \, dx}_{\overset{\shortparallel}{\langle F, v\rangle}} \;\; \forall v \in V_0$

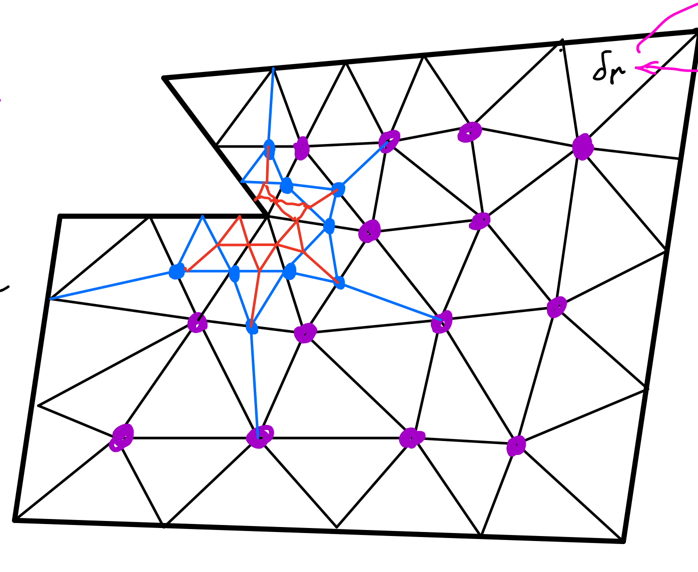
$$a(u,v) = \langle F, v \rangle \;\; \forall v \in V_0$$

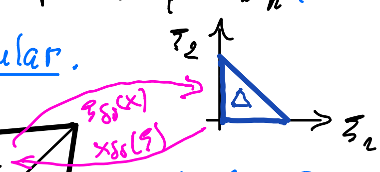
has a unique solution $u \in V_0$ due to Lax-Milgram.

- Let $\mathcal{T}_h = \{\delta_r : r \in R_h\}$ be a triangulation of Ω such that <u>Def. 3.3: (8)-(10)</u> resp. (4) holds with h replaced by the local mesh size $h_r = h^{(r)}$, $r \in R_h$. Such meshes are called <u>shape regular</u>.



$$1 \le \frac{\varrho_0^{(r)}}{\varrho_i^{(r)}} \le c_\Sigma$$





Def. 3.3.:

(8) $\underline{c}_1 h_r^2 \le |J_{\delta_r}| \le \bar{c}_1 h_r^2$

(9) $\| J_{\delta_r} \| \le c_2 h_r$

(10) $\| J_{\delta_r}^{-T} \| \le c_3 h_r^{-1}$

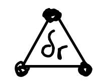
$$J_{\delta_r} = \frac{\partial x_{\delta_r}(\xi)}{\partial \xi}$$

Jacobian

$\mathcal{T}_h = \{\delta_r : r \in R_h\}$ - shape regular, $\quad \omega_h = \{\bullet\}$.

- We now define the corresponding to the shape regular triangulation $\mathcal{T}_h = \{\delta_r : r \in R_h\}$ FE spaces

$$V_{0h} := \operatorname{span}\{p^{(i)} : i \in \omega_h\} \subset V_0$$



based on conforming linear triangular (Courant's) elements ($S(\Delta) = P_1(\Delta)$). Then we obtain the FE GALERKIN scheme

$(45)_h$ Find $u_h \in V_{0h}$: $a(u_h, v_h) = \langle F, v_h \rangle \;\forall v_h \in V_{0h}$.

- We now look for a <u>computable</u> and <u>localizable</u> upper bound

$$\eta(u_h) := \sqrt{\sum_{r \in R_h} \eta^2_{\delta_r}(u_h)}$$

such that there exist positive generic constants $\underline{c} = \underline{c}(\Omega, c_1, c_2, c_3) = \underline{c}(c_r)$ and $\overline{c} = \overline{c}(\Omega, c_1, c_2, c_3) = \overline{c}(c_r)$:

1. $\|u - u_h\|_1 \le \overline{c}\,\eta(u_h)$ (reliability)

2. $\underline{c}\,\eta(u_h) \le \|u - u_h\|_1$ (efficiency)

If 1. and 2. are valid, then the a posteriori error estimator is <u>reliable</u> and <u>efficient</u>.

The quantity

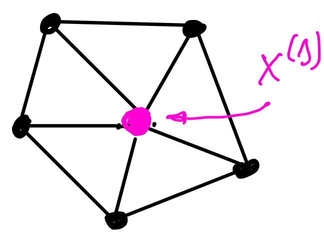
$$1 \le I_{eff} := \frac{\overline{c}\,\eta(u_h)}{\|u - u_h\|_1} \le \frac{\overline{c}}{\underline{c}}$$

is called <u>efficiency index</u>.

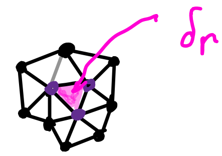
3.6.1. The CLÉMENT Interpolator

- **Notations:** neighbourhoods

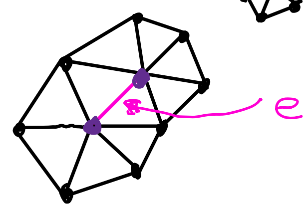
$$\bar{U}(x^{(j)}) := \bigcup_{r \in B_j := \{ r \in \mathbb{R}_h :\, x^{(j)} \in \bar{\delta}_r \}} \bar{\delta}_r$$



$$\bar{U}(\delta_r) := \bigcup_{x^{(j)} \in \bar{\delta}_r} \bar{U}(x^{(j)})$$



$$\bar{U}(e) := \bigcup_{x^{(j)} \in \bar{e}} \bar{U}(x^{(j)})$$



- **Def. 3.17:** CLÉMENT Interpolator

The operator $I_h : L_2(\Omega) \longrightarrow V_h = \text{span}\{ p^{(i)} : i \in \bar{\omega}_h \}$ which is defined by the relation

$$I_h v(x) := \sum_{j \in \bar{\omega}_h} \underbrace{(P_j v)}_{\in \mathbb{R}^1 = \mathbb{P}_0 = \text{space of polynomials of the degree } 0} p^{(j)}(x) \in V_h \quad \forall v \in L_2(\Omega)$$

is called CLÉMENT interpolator, where, for every grid point $x^{(j)}$, $j \in \bar{\omega}_h$, we define the local L_2-projection

$$P_j : L_2(U(x^{(j)})) \longrightarrow \mathbb{P}_0(U(x^{(j)})) = \mathbb{R}^1 :$$

$$\int_{U(x^{(j)})} P_j v \cdot c \, dx = \int_{U(x^{(j)})} v(x) \cdot c \, dx \quad \forall c \in \mathbb{P}_0(U(x^{(j)})),$$

i.e.
$$P_j v = \frac{1}{|U(x^{(j)})|} \int_{U(x^{(j)})} v(x)\, dx$$
~ patch avarage of v, where $|U(x^{(j)})| = \int_{U(x^{(j)})} dx$.

Remark:

For $u \in H^1(\Omega)$, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, the Lagrange interpolator

$$I_h v(x) := \mathrm{int}_{V_h}(v)(x) := \sum_{j \in \overline{\omega}_h} v(x^{(j)})\, p^{(j)}(x)$$

is NOT defined since $H^1(\Omega) \not\subset C(\overline{\Omega})$,
BUT CLÉMENT is defined!

Lemma 3.18:

Ass.: Let $\mathcal{T}_h$ be a shape regular triangulation of $\Omega \subset \mathbb{R}^2$, and $V_h := \mathrm{span}\{p^{(i)} ; i \in \overline{\omega}_h\}$ the corresponding Courant's FE space (Lin. tri. el.: $\triangle$).

St.: Then CLÉMENT's interpolation operator I_h fulfills the following error estimates:

1) $\|v - I_h v\|_{k,\delta_r} := \|v - I_h v\|_{H^k(\delta_r)} \leq c\, h_r^{1-k} |v|_{1,U(\delta_r)}$, $\forall r \in \mathbb{R}_h$ $\forall v \in H^1(\Omega)$, $k = 0, 1$,

2) $\|v - I_h v\|_{0,e} := \|v - I_h v\|_{L_2(e)} \leq c\, h_e^{1/2} |v|_{1,U(\delta_r)}$, $\forall e \in \mathcal{E}_i := \{\text{internal edges}\}$, $\forall v \in H^1(\Omega)$,

where $h_r = h^{(r)}$, $h_e = |e|$, $c = \mathrm{const} > 0$, $c \neq c(r, h)$.

Proof: 1) $k=1$ and 2) mms*

Sketch of the proof for 1)$_{k=0}$: $\|v - I_h v\|_{0,\delta_r} \leq c\, h_r |v|_{1,U(\delta_r)}$

- $\mathcal{T}_h$ ~ shape regular $\Rightarrow$ $|U(\delta_r)| := \mathrm{meas}\, U(\delta_r) \leq \tilde{c} h_r^2$ (mms)
- Bramble-Hilbert-Lemma 2.17 (Poincaré) yields (mms):

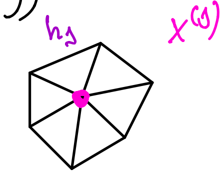
$$\|v - P_j v\|_{0,U(x^{(j)})} \leq c\, h_j |v|_{1,U(x^{(j)})} \quad \forall v \in H^1(U(x^{(j)}))$$

L21-07

BH $\quad \| v - P_j v \|_{L_2(U(x^{(j)}))} \le c\, h_j\, |v|_{H^1(U(x^{(j)}))}$

where $c = \mathrm{const} > 0,\ c \ne c(h),$

$$h_j = \max_{r:\ \delta_r \subset U(x^{(j)})} h_r \approx h_r = h^{(r)}$$



Poincaré: L07-10

$$\int_{U(x^{(j)})} \Big| v - \frac{1}{|U(x^{(j)})|} \int_{U(x^{(j)})} v(x)\,dx \Big|^2 dx \le c_p^2 \int_{U(x^{(j)})} |\nabla v|^2 dx$$

$U(x^{(j)})$ – convex: $c = 1/\pi$

$$c_p^2 \le c^2 (\mathrm{diam}\, U(x^{(j)}))^2 \approx c^2 h_j^2$$

- $$\| v - I_h v \|_{0,\delta_r}^2 = \int_{\delta_r} \Big[v(x) - \sum_{j \in \bar\omega_h} (P_j v)\, p^{(j)}(x) \Big]^2 dx$$

$j \in \bar\omega_h \to j \in \bar\omega_r = \{ j \in \bar\omega_h : x^{(j)} \in \bar\delta_r \}$

$$= \int_{\delta_r} \Big[\overbrace{\sum_{j \in \bar\omega_r} 1 \cdot p^{(j)}(x)}^{=1 \text{ on } \delta_r} v(x) - \sum_{j \in \bar\omega_r} (P_j v)\, p^{(j)}(x) \Big]^2 dx$$

$$= \int_{\delta_r} \Big[\sum_{j \in \bar\omega_r} p^{(j)}(x) \big(v(x) - P_j v \big) \Big]^2 dx$$

Σ Cauchy

$$\le \int_{\delta_r} \sum_{j \in \bar\omega_r} (p^{(j)}(x))^2 \sum_{j \in \bar\omega_r} (v(x) - P_j v)^2 dx$$

$$\le \underbrace{\max_{x \in \bar\delta_r} \sum_{j \in \bar\omega_r} (p^{(j)}(x))^2}_{\le 3} \sum_{j \in \bar\omega_r} \int_{\delta_r} |v(x) - P_j v|^2 dx$$

$\delta_r \subset U(x^{(j)})$, $j \in \bar\omega_r$

$$\le 3 \sum_{j \in \bar\omega_r} \int_{U(x^{(j)})} |v(x) - P_j v|^2 dx$$

$$\le 3 c^2 \sum_{j \in \bar\omega_r} h_j^2 |v|^2_{1,U(x^{(j)})} \le 3 \bar c^2 h_r^2 |v|^2_{1,U(\delta_r)} .$$

q.e.d.

Remark:

1. $$|v - I_h v|_{k,\Omega}^2 \simeq \sum_r |v - I_h v|_{H^k(\delta_r)}^2 \overset{\text{L.3.18}}{\leq} c^2 \sum_r h_r^{2(1-k)} |v|_{1,U(\delta_r)}^2 \lesssim \tilde{c}^2 h^{2(1-k)} |v|_{1,\Omega}^2$$

2. $k=1$: $$|v - I_h v|_{1,\Omega} \leq c h^0 |v|_{1,\Omega}$$
$$\Leftrightarrow |I_h v|_{1,\Omega} \leq |v|_{1,\Omega} + |I_h v - v|_{1,\Omega} \leq (1 + c h^0) |v|_{1,\Omega}$$
$$\Leftrightarrow I_h \in L(H^1, H^1)$$

3. $k=0$: $$|v - I_h v|_{0,\Omega} \leq c_0 h^1 |v|_{1,\Omega}$$
$k=1$: $$|v - I_h v|_{1,\Omega} \leq c_0 h^0 |v|_{1,\Omega}$$

Space interpolation yields:

$k=s$: $$|v - I_h v|_{H^s(\Omega)} \leq c_s h^{1-s} |v|_{1,\Omega}$$

for $s \in (0,1)$, i.e., we have it for $s \in [0,1]$.