3.5.3. The Second Lemma of STRANG

- Consider now the following situation:

(1) Find $u \in \bar{V}_g$: $a(u,v) = \langle F, v \rangle \quad \forall v \in \bar{V}_0 \subset \bar{V}$

$\tilde{K}_h \tilde{\underline{u}}_h = \tilde{\underline{f}}_h$

$(\tilde{1})_h$ Find $\tilde{u}_h \in \bar{V}_{gh}$: $a_h(\tilde{u}_h, v_h) = \langle F_h, v_h \rangle_h \quad \forall v_h \in \bar{V}_{0h} \subset \bar{V}_h$

$V_{gh} = g_h + \bar{V}_{0h}$ Galerkin-Petrov $\nearrow$ $\bar{U}_{0h} \subset \bar{U}_h$

under the <u>standard assumptions</u> (33) and the additional assumptions (39):

(39)
1) $|\langle F_h, v_h \rangle_h| \le \|F_h\|_{V_{0h}^*} \|v_h\|_h \quad \forall v_h \in \bar{V}_{0h}$

$\leftarrow$ (discrete) norm in $\bar{V}_{0h}$!

2) $a_h(\cdot,\cdot) : \bar{V}_h \times \bar{V}_h \to \mathbb{R}^1$ – discrete bilinear form:

2a) $\exists \mu_3 = \text{const} > 0$, $\mu_3 \neq \mu_3(h)$:

$a_h(v_h, v_h) \ge \mu_3 \|v_h\|_h^2 \quad \forall v_h \in \bar{V}_{0h}$,

2b) $\exists \mu_4 = \text{const} > 0$, $\mu_4 \neq \mu_4(h)$:

$|a_h(v, v_h)| \le \mu_4 \|v\|_{*h} \|v_h\|_h \quad \forall v_h \in \bar{V}_{0h}$,

$\forall v \in \{(g + \bar{V}_0) \cap H\} - \bar{V}_{gh} \subset (\bar{V} \cap H) + \bar{V}_h$,

where $H \subset \bar{V}$ is a suitable subspace s.t. $u \in \bar{V}_g \cap H$, and the "discrete" norm $\|\cdot\|_{*h}$ is defined on $(\bar{V} \cap H) + \bar{V}_h$.

e.g. $\|\cdot\|_h = \|\cdot\|_{*h}$, or $\|\cdot\|_h \le \|\cdot\|_{*h}$ stronger norm
think about $\bar{V} = H^1 \supset H^2 = H$

- **Remark:** 1. This situation is typical for

a) non-conforming FE ($V_h \not\subset V$), e.g. dG (Chapter 4) !

b) the approximation of essenial BC: $V_{0h} \not\subset \bar{V}_0$ and or $\bar{V}_{gh} \not\subset \bar{V}_g$!

c) GALERKIN-PETROV: $v_h \in \bar{U}_{0h} \subset \bar{U}_0$, $\dim \bar{V}_{0h} = \dim \bar{U}_{0h}$, e.g. SUPG

2. Add. Ass. $(39)^{2a)}$ ensures $\exists! \, \tilde{u}_h \in \bar{V}_{gh} : (\tilde{1})_h$. Proof: $! \Rightarrow \exists$ $(\dim \bar{V}_h < \infty)$

Theorem 3.16: (The 2nd Lemma of Strang)

Ass.: Let $u \in V_g \cap H$ (e.g. $V_g \subset H^1$, $H = H^2$), and let the standard assumptions (33) and the additional assumption (34) be fulfilled. In general, we do NOT assume that $V_h \subset V$, $V_{0h} \subset V_0$, $V_{gh} = g_h + V_{0h} \subset V_g$ ($g_h \in V_g$).

St.: Then the discretization error estimates

$$\text{(40)} \quad \|u - \tilde{\tilde{u}}_h\|_h \le \inf_{v_h \in V_{gh}} \left[\|u - v_h\|_h + \frac{\mu_4}{\mu_3} \|u - v_h\|_{*h} \right] + \frac{1}{\mu_3} \sup_{w_h \in V_{0h}} \frac{|a_h(u, w_h) - \langle F_h, w_h \rangle_h|}{\|w_h\|_h}$$

$\|\cdot\|_h \le \|\cdot\|_{*h}$

$$\le \left(1 + \frac{\mu_4}{\mu_3}\right) \underbrace{\inf_{v_h \in V_{gh}} \|u - v_h\|_{*h}}_{\text{approximation error}} + \underbrace{\frac{1}{\mu_3} \sup_{w_h \in V_{0h}} \frac{|a_h(u, w_h) - \langle F_h, w_h \rangle_h|}{\|w_h\|_h}}_{\text{consistency error}}$$

hold.

Proof: Ass. $u \in V_g \cap H$ means add. regularity like $H = H^{1+s}$, $V_g \subset H^1$.

- Let $v_h \in V_{gh}$ be an arbitrary element (function) from V_{gh}. Using triangle inequality, we obtain

$$\text{(41)} \quad \|\underbrace{u - \tilde{\tilde{u}}_h}_{\in (V_g \cap H) - V_{gh} \subset (V \cap H) - V_{gh}}\|_h \le \|u - v_h\|_h + \|\underbrace{\overset{V_{gh}}{v_h} - \overset{\in V_{gh} = g_h + V_{0h}}{\tilde{\tilde{u}}_h}}_{=: -w_h \in V_{0h} \text{ - arbitrary}}\|_h .$$

- For $w_h = \tilde{\tilde{u}}_h - v_h \in \bar{V}_{0h}$, we now obtain the estimates

(42) $$\mu_3 \| \tilde{\tilde{u}}_h - v_h \|_h^2 \underset{(39)_{2a)}}{\leq} a_h(\tilde{\tilde{u}}_h - v_h, \underbrace{u_h - v_h}_{= w_h}) = a_h(\tilde{\tilde{u}}_h \overset{-u+u}{\vee} - v_h, w_h)$$

$$\overset{-u+u}{=} a_h(u - v_h, w_h) + [\, a_h(\tilde{\tilde{u}}_h, w_h) - a_h(u, w_h)\,]$$

$$a_h(\tilde{\tilde{u}}_h, w_h) = \langle F_h, w_h \rangle_h \leftarrow (\tilde{\tilde{1}})_h$$

$$= a_h(\underbrace{u - v_h}_{\in (\bar{V}_g \cap H) - \bar{V}_{gh}\ !}, w_h) + [\, \langle F_h, w_h\rangle_h - a_h(u, w_h)\,]$$

$$\underset{(39)_{2b)}}{\leq} \mu_4 \| u - v_h \|_{*h} \| w_h \|_h + [\, \langle F_h, w_h \rangle_h - a_h(u, w_h)\,]$$

- We again divide (42) by $\| w_h \|_h = \| u_h - v_h \|_h$ and μ_3, and then we take the supremum over all $w_h \in \bar{V}_{0h}$ of the right-hand side of this inequality:

(43) $$\| \tilde{\tilde{u}}_h - v_h \|_h \leq \frac{\mu_4}{\mu_3} \| u - v_h \|_{*h} + \frac{1}{\mu_3} \sup_{w_h \in \bar{V}_{0h}} \frac{| a_h(u, w_h) - \langle F_h, w_h \rangle_h |}{\| w_h \|_h}$$

- (41), (43) and $\inf\limits_{v_h \in \bar{V}_{gh}}$ immediately yield (40). Indeed,

$$\| u - \tilde{\tilde{u}}_h \|_h \leq \inf_{v_h \in \bar{V}_{gh}} \left[\| u - v_h \|_h + \frac{\mu_4}{\mu_3} \| u - v_h \|_{*h} \right] + \frac{1}{\mu_3} \sup_{w_h \in \bar{V}_{0h}} \frac{| a_h(u, w_h) - \langle F_h, w_h \rangle_h |}{\| w_h \|_h}$$

Ass. $\| u - v_h \|_h \leq \| u - v_h \|_{*h}$

$$\leq \left(1 + \frac{\mu_4}{\mu_3}\right) \inf_{v_h \in \bar{V}_{gh}} \| u - v_h \|_{*h} + \frac{1}{\mu_3} \sup_{w_h \in \bar{V}_{0h}} \frac{| a_h(u, w_h) - \langle F_h, w_h \rangle_h |}{\| w_h \|_h}.$$

q.e.d

■ **Remark:** see also Chapter 4, Sect. 4.1. dG Schemes!

The discrete scheme: Find $\tilde{\tilde{u}}_h \in \bar{V}_{gh}$:

$$a_h(\tilde{\tilde{u}}_h, v_h) = \langle F_h, v_h \rangle_h \quad \forall v_h \in \bar{V}_{0h}$$

is called consistent if

$$a_h(u, v_h) = \langle F_h, v_h \rangle_h \quad \forall v_h \in \bar{V}_{0h}.$$

This implies that the consistency error in (40) is 0, and furthermore we get "discrete" Galerkin orthogonality:

$$a_h(u - \tilde{\tilde{u}}_h, v_h) = 0 \quad \forall v_h \in \bar{V}_{0h}.$$

Therefore, discretization error estimate (40) takes the form:

$$\| u - \tilde{\tilde{u}}_h \|_h \le \inf_{v_h \in \bar{V}_{gh}} \Big[\| u - v_h \|_h + \frac{\mu_4}{\mu_3} \| u - v_h \|_{*h} \Big]$$

$$\le \Big(1 + \frac{\mu_4}{\mu_3} \Big) \inf_{v_h \in \bar{V}_{gh}} \| u - v_h \|_{*h}.$$

Ass.: $\| u - v_h \|_h \le \| u - v_h \|_{*h}$