3.5. Convergence Analysis in the Non-standard Case

3.5.1. Violation of the Variational Principle (Variational Crimes)

Standard Case (Variational Principle = Galerkin Principle)

$$(1) \quad \text{Find } u \in V_g: \ a(u,v) = \langle F, v\rangle \quad \forall v \in V_0 \subset V$$

$$\cup \qquad\qquad\qquad\qquad \cup \quad \cup$$

$$(1)_h \quad \text{Find } u_h \in V_{gh}: \ a(u_h,v_h) = \langle F, v_h\rangle \quad \forall v_h \in V_{0h} \subset V_h$$

$\boxed{3}$ $\boxed{5}$ $\boxed{4}$ $\boxed{2}$ $\boxed{1}$

+ Standard Assumptions: $\boxed{6}$

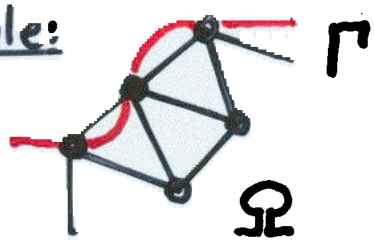
$$(33) \quad \begin{cases} 1) & F \in V_0^*, \\ 2) & a(\cdot,\cdot): V\times V \to \mathbb{R}^1 \sim \text{continous b.f.}: \\ 2a) & a(v,v) \geq \mu_1 \|v\|^2 \quad \forall v \in V_0, \\ 2b) & |a(u,v)| \leq \mu_2 \|u\| \|v\| \quad \forall u,v \in V_0. \end{cases}$$

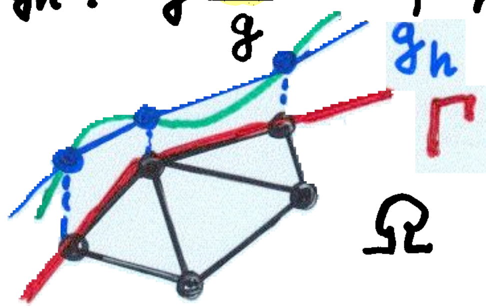
■ **The practice often forces us to violate the standard approach (= variational principle):**
$\Rightarrow$ **Variational Crimes:**

1. **Numerical Integration:** $\int \longrightarrow \sum$
   $\mathrm{Jn}\,(1)_h$: $a(\cdot,\cdot) \longrightarrow a_h(\cdot,\cdot)$ $\boxed{5}$ $\curvearrowright \widetilde{(1)}_h$
   $\langle F, \cdot\rangle \longrightarrow \langle F_h, \cdot\rangle_h$ $\boxed{\cancel{4}}$

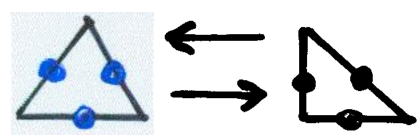
2. 1st Kind BC cannot always be fulfilled precisely in V_h!
   i.e. $V_{0h} \not\subset V_0$ $\boxed{\cancel{2}}$ and/or $V_{gh} \not\subset V_g$ $\boxed{\cancel{3}}$ even if $V_h \subset V$!

   Example:

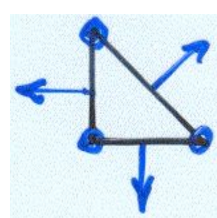




3. **Conformity of the elements is violated:**
   (in particular, for 4th order PDEs: $V = H^2(\Omega)$!),
   i.e. $V_h \not\subset V$ $\boxed{1}$ (e.g. the use of C^0-el. for 4th order PDEs)
   Examples: a) 2nd order PDEs: Crouzeix-Raviart-el.



   b) 4th orde PDEs: Morley-element
   $\mathcal{F}(\Delta) = P_2$



4. **Assumptions $(33)_{2a)+2b)}$ $\boxed{6}$ have to be completed by additional conditions imposed on the discrete bilinear form $a_h(\cdot,\cdot)$**
   $\Rightarrow$ (34) resp. (39)!

3.5.2. The First Lemma of STRANG

- We first consider the following situation:

(1) Find $u \in V_g : a(u,v) = \langle F, v \rangle = \ell(v) \quad \forall v \in V_0 \subset V$

$(\tilde{1})_h$ Find $\tilde{u}_h \in V_{gh} : a_h(\tilde{u}_h, v_h) = \langle F_h, v_h \rangle_h = \ell_h(v_h) \quad \forall v_h \in V_{0h} \subset V_h$

$(\underset{\sim}{\tilde{1}})_h$ $\quad \cancel{\boxed{5}} \quad \tilde{K}_h \underset{\sim}{\tilde{u}}_h = \underset{\sim}{\tilde{f}}_h \quad \cancel{\boxed{4}}$

with $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}^1$ – continous discrete bilinear form,

$\langle F_h, \cdot \rangle_h = \ell_h(\cdot) : V_{0h} \to \mathbb{R}^1$ – continous linear form,

under the <u>standard assumptions</u> (33) and

under the <u>additional assumptions</u> (34) of the uniform V_{0h}-ellipticity of the discrete bilinear form $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}^1$:

(34) $\exists \mu_3 = \text{const} > 0 : \; a_h(v_h, v_h) \geq \mu_3 \|v_h\|_V^2 \quad \forall v_h \in V_{0h} \; \forall h \in \Theta$

This situation is typical for <u>numerical integration</u>! $(\downarrow)$

<u>Remark:</u> $\exists! \, \tilde{u}_h \in V_{gh} : (\tilde{1})_h$ follows from (34)! $\quad (! \Rightarrow \exists)$

- <u>**Theorem 3.15:**</u> (The 1st Lemma of Strang)

Ass.: Let $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}^1$ – cont. bil. form, $F_h \in V_{0h}^*$,

$V_h \subset V$, $V_{gh} \subset V_g$, $V_{0h} \subset V_0$, and let Ass. (33)–(34) be fulfilled.

<u>St.:</u> Then the discretization error estimate

$$(35) \quad \|\overset{(1)}{u} - \overset{(\tilde{1})_h}{\tilde{u}_h}\| \leq c \left[\inf_{v_h \in V_{gh}} \left\{ \|u - v_h\| + \sup_{w_h \in V_{0h}} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} \right\} + \sup_{w_h \in V_{0h}} \frac{|\langle F, w_h \rangle - \langle F_h, w_h \rangle_h|}{\|w_h\|} \right]$$

holds, with

$c = \max\left\{ \left(1 + \frac{\mu_2}{\mu_3}\right), \frac{1}{\mu_3} \right\}$.

Proof:

- Let $v_h \in \bar{V}_{gh}$ be an arbitrary FE function. Then

$$(36)\qquad \|u - \tilde{u}_h\| \le \|u - v_h\| + \underbrace{\|v_h - u_h\|}_{=:\,-\,w_h \in \bar{V}_{0h}\,-\,\text{arbitrary!}}$$

- For $w_h = u_h - v_h \in \bar{V}_{0h}$, we obtain the estimates

$$(37)\qquad \mu_3 \|u_h - v_h\|^2 \le a_h(u_h - v_h, u_h - v_h) = a_h(u_h - v_h, w_h) =$$

$$= a(u - v_h, w_h) + [a(v_h, w_h) - a_h(v_h, w_h)] + [\underbrace{a_h(u_h, w_h)}_{\overset{(\tilde{1})_h}{=}\langle F_h, w_h\rangle_h} - \underbrace{a(u, w_h)}_{\overset{(1)}{=}\langle F, w_h\rangle}]$$

$$= a(u - v_h, w_h) + [a(v_h, w_h) - a_h(v_h, w_h)] + [\langle F_h, w_h\rangle_h - \langle F, w_h\rangle]$$

$$\overset{(33)}{\le} \mu_2 \|u - v_h\|\,\|w_h\| + [a(v_h, w_h) - a_h(v_h, w_h)] + [\langle F_h, w_h\rangle_h - \langle F, w_h\rangle]$$

- First we divide (37) by $\|w_h\| = \|u_h - v_h\|$ and μ_3, and then we take the supremum over all $w_h \in \bar{V}_{0h} \setminus \{0\}$ of the right-hand side of this inequality:

$$(38)\qquad \|u_h - v_h\| \le \frac{\mu_2}{\mu_3}\|u - v_h\| + \frac{1}{\mu_3} \sup_{w_h \in \bar{V}_{0h}\setminus\{0\}} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} +$$
$$+ \frac{1}{\mu_3} \sup_{w_h \in \bar{V}_{0h}\setminus\{0\}} \frac{|\langle F, w_h\rangle - \langle F_h, w_h\rangle_h|}{\|w_h\|}$$

- (36), (38) and $\inf\limits_{v_h \in \bar{V}_{gh}}$ immediately yield (35). Indeed,

$$\|u - u_h\| \le \inf_{v_h \in \bar{V}_{gh}} \left\{ \left(1 + \frac{\mu_2}{\mu_3}\right)\|u - v_h\| + \frac{1}{\mu_3} \sup_{w_h \in \bar{V}_{0h}} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} + \right.$$
$$+ \frac{1}{\mu_3} \sup_{w_h \in \bar{V}_{0h}} \frac{|\langle F, w_h\rangle - \langle F_h, w_h\rangle_h|}{\|w_h\|}$$

q.e.d.

The typical application of the 1st Lemma of Strang = Analysis of the effects of numerical integration:

Let us again consider the model problem from Subsection 3.2.1 with data $a = 0$, $\varkappa = 0$, $g_1 = g_2 = g_3 = 0$:

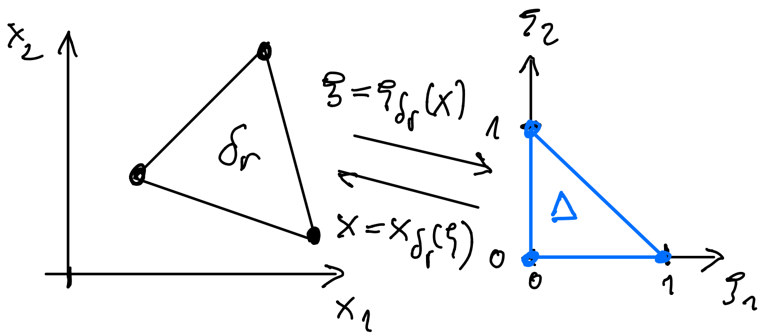
- $V = H^1(\Omega)$, $V_0 = \{ v \in V : v = 0 \text{ on } \Gamma_1 \} = V_g$,

- $$a(u,v) = \int_\Omega \lambda(x) \nabla_x u(x) \cdot \nabla_x v(x)\, dx = \sum_{r \in R_h} \int_{\delta_r} \lambda(x) \nabla_x u(x) \cdot \nabla_x v(x)\, dx$$

$x = x_{\delta_r}(\xi)$, $\nabla_x = J_{\delta_r}^{-T} \nabla_\xi$

$$a_h(u_h, v_h) = \sum_{r \in R_h} \sum_{\beta \in I_r} w_r^{(\beta)} \left[\, \bullet \,\right]_{\xi = \xi_r^{(\beta)}} \cdot \underbrace{\tfrac{1}{2}}_{= \text{meas } \Delta}$$

weights; integr. points $\longrightarrow$ $\xi = \xi_r^{(\beta)}$



Subsection 3.2.4

$S(\Delta) = P_1$
$I_r = I = \{1\}$
$\xi_r^{(1)} = (\tfrac{1}{3}, \tfrac{1}{3})$
$w_r^{(1)} = 1$

where
$$\left[\, \bullet \,\right] = \lambda(x_{\delta_r}(\xi))\, J_{\delta_r}^{-T} \nabla_\xi u_h(x_{\delta_r}(\xi)) \cdot J_{\delta_r}^{-T} \nabla_\xi u_h(x_{\delta_r}(\xi))\, |J_{\delta_r}|$$
$$= \lambda(x_{\delta_r}(\xi))\, \nabla_\xi^T u_h(\cdot)\, J_{\delta_r}^{-1} J_{\delta_r}^{-T} \nabla_\xi u_h(\cdot)\, |J_{\delta_r}(\xi)|.$$

- $$\langle F, v \rangle = \int_\Omega f(x) v(x)\, dx = \sum_{r \in R_h} \int_{\delta_r} f(x) v(x)\, dx$$

$$\langle F_h, v_h \rangle = \sum_{r \in R_h} \sum_{\beta \in I_r} w_r^{(\beta)} f(x_{\delta_r}(\xi_r^{(\beta)}))\, v_h(x_{\delta_r}(\xi_r^{(\beta)}))\, |J_{\delta_r}(\xi_r^{(\beta)})|$$

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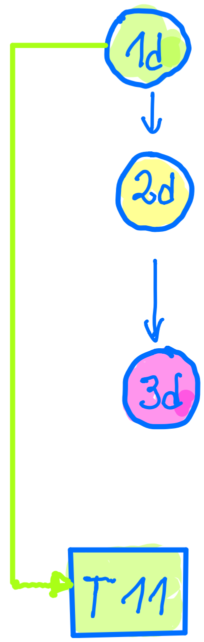
Ex. 3.13 Formulate conditions imposed on the data $\lambda \in L_\infty(\Omega) \cap W_2^2(\delta_r)$, $f \in L_2(\Omega) \cap W_2^2(\delta_r)$ $\forall r \in R_h$ $\forall h \in \Theta$ and on the quadrature rule (i.e. algebraic exactness for $\int_\Delta \varphi(\xi)\, d\xi$) such that

$$\| u - \tilde{u}_h \|_{1,\Omega} \le c(u,f,\lambda)\, h,$$

where $\tilde{u}_h$ solves $(\tilde{1})_h$!

Can you already ensure the $O(h)$ accuracy for Δ $(S(\Delta) = P_1,\ I_r = \{1\},\ w_r^{(1)} = 1,\ \xi_r^{(1)} = (1/3, 1/3))$!



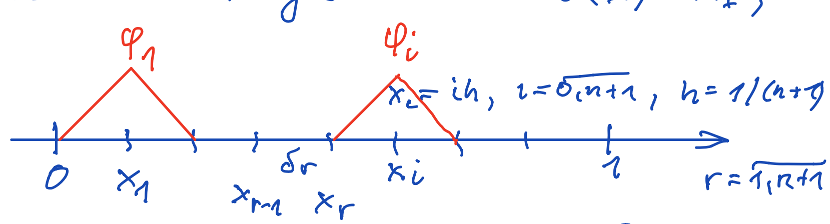
Let us conside the 1d BVP: Find $u \in V_g = V_0 = \overset{\circ}{H}{}^1(0,1)$:

$$(1) \qquad \underbrace{\int_0^1 \lambda(x)\, u'(x)\, v'(x)\, dx}_{a(u,v)} = \underbrace{\int_0^1 f(x)\, v(x)\, dx}_{\langle F, v \rangle} \quad \forall v \in V_0$$

where $f \in L_2(0,1)$ and $\lambda \in L_\infty(0,1)$:

$\exists\ \underline{\lambda}, \overline{\lambda} = \text{const} > 0: \quad 0 < \underline{\lambda} \le \lambda(x) \le \overline{\lambda} \quad \forall x \in (0,1).$

FE-discretization by Linear FE: $S(\Delta) = P_1$,



$V_{gh} = V_{0h} = \text{span}\{\varphi_1, \ldots, \varphi_n\} \subset V_0 = \overset{\circ}{H}{}^1(0,1)$,

$$(1)_h \qquad \text{Find } u_h \in V_{0h}: \ a(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in V_{0h}.$$

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Now we approximate the bilinear form $a(\cdot,\cdot)$ and the Linear form $\langle F, \cdot \rangle$ defined in (1) using numerical integration, namely the ==mid point rule==: GAUSS 1

$$a_h(u_h, v_h) = \sum_{r=1}^{n+1} h_r \lambda(x_r^*)\, u_h'(x_r^*)\, v_h'(x_r^*),$$

$$\langle F_h, v_h \rangle_h = \sum_{r=1}^{n+1} h_r f(x_r^*)\, v_h(x_r^*),$$

with $x_r^* = x_{\delta_r}(\frac{1}{2}) = x_{r-1} + \frac{1}{2} h$. To ensure that these expressions are well-defined, we have to assume that $\lambda, f \in C[0,1]$.

In practice, we now solve the problem

$(\tilde{1})_h$ Find $\tilde{u}_h \in \overset{\circ}{V}_{0h}$: $a_h(\tilde{u}_h, v_h) = \langle F_h, v_h \rangle_h \; \forall v_h \in \overset{\circ}{V}_{0h}$.

$$\tilde{K}_h \underline{\tilde{u}}_h = \underline{\tilde{f}}_h$$

<u>Questions:</u> 1. Has problem $(\tilde{1})_h$ a unique solution?

2. Under which conditions imposed on the data $\lambda(\cdot)$ and $f(\cdot)$ the discretization error $|u - \tilde{u}_h|_{H^1(0,1)}$ has the same order $O(h)$ as $|u - u_h|_{H^1(0,1)}$ (exact finite element scheme)?

$\|\cdot\|_{\overset{\circ}{V}_0} := |u|_{H^1(0,1)}$

seminorm in H^1 is norm in $\overset{\circ}{H}^1$

$\|\cdot\|_V \simeq \|\cdot\|_{V_0}$

Answer: is given by the 1st Lemma of STRANG!

① Standard Assumptions (33) ✓

② Additional Assumption (34): $\exists \mu_3 = const > 0$:

$\exists!$ $\Leftarrow$ $$a_h(v_h, v_h) \geq \mu_3 |v_h|^2_{H^1(0,1)} \quad \forall v_h \in V_{0h}.$$

$\|\cdot\| = |\cdot|_1 := |\cdot|_{H^1}$

③ Th. 3.15: $|u - \tilde{u}_h|_1 \leq c \left(\inf_{v_h \in \tilde{V}_{gh}} \{ |u - v_h|_1 + \dots \right)$

$v_h = u_h \Rightarrow \inf_{v_h \in \tilde{V}_{gh} = V_{0h}} |u - v_h|_1 \leq |u - u_h|_1 \leq c\, h\, |u|_2$

Th. 3.8

④ $|a(u_h, w_h) - a_h(u_h, w_h)| \leq c(u, \lambda)\, h\, |w_h|_1$

Ass. imposed on λ? $\quad \forall w_h \in \tilde{V}_{0h}$

⑤ $|\langle F, w_h \rangle - \langle F_h, w_h \rangle_h| \leq c(f)\, h\, |w_h|_1$

Ass. imposed on f

Then Theorem 3.15 gives estimate

(35) $$\boxed{|u - \tilde{u}_h|_{H^1(0,1)} \leq c(u, \lambda, f)\, h}$$