3.4.5. L_∞- and W^1_∞ - Convergence

- Under the Assumptions of Theorm 3.10 (L_2-conv.), we will prove the **L_∞ - error estimate**

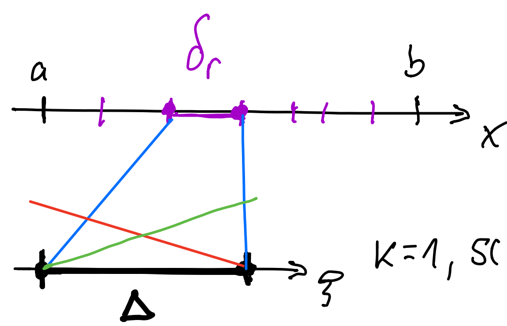
$$(28)\quad \underbrace{\|u-u_h\|_{0,\infty,\Omega}}_{\in C(\bar\Omega)} = \max_{x\in\bar\Omega} |u(x)-u_h(x)| \le c\, h^{k+1-\frac{d}{2}} |u|_{k+1,\Omega}$$

$L_\infty(\Omega)$ generic constant $2(k+1)>d$ $W_2^{k+1}(\Omega) \hookrightarrow C(\bar\Omega)$

with some positive constant $c = const > 0$, $c \neq c(u,h)$.

- Example:

$d=1$: $\Omega = (a,b)$



$\bar\Delta = [0,1]$ $\quad K=1,\ S(\Delta)=P_1$

Embedding in 1d: $W_2^1(a,b) \hookrightarrow C[a,b]$

$$\Longrightarrow \|u-u_h\|_{C[a,b]} \le c_E \|u-u_h\|_{H^1(a,b)} \le c_E c_{1,2}\, h^1 |u|_{2,\Omega}$$

BUT a) $(28) \Rightarrow h^{3/2}$ if $u \in W_2^2(a,b) = H^2(a,b)$!

b) If $u \in W^2_\infty(a,b)$, then

$$2\left[\begin{array}{ll} \|u-u_h\|_{C[a,b]} = O(h^2) & (\text{mms}^*)\ ! \\ \|u - Int_{\bar V_h}(u)\|_{C[a,b]} = O(h^2) & (\text{mms})\ ! \end{array}\right.$$

$d=2,3$: Embedding $W_2^1(\Omega) \hookrightarrow C(\bar\Omega)$ is not true, i.e

L_∞-error estimates are not trivial !

There is no L_∞-Cea-Like Lemma in general !

- We first prove so-called **inverse inequalities**
  [= Markov (brother's) inequalities (1890)
  = Bernstein inequalities (1922)]
  which are very useful in our numerical analysis:

Lemma 3.11: (some important inverse inequalities)

Ass.: Let Ass. 1 (regular triangulation) of the approximation Theorem 3.6 be fulfilled, and Let $\dim S(\Delta) = |A| < \infty$, $S(\Delta) = \mathrm{span}\{p^{(\alpha)} : \alpha \in A\}$.

St.: Then the following inequalities are valid:

a) $\|v_h\|_{1,\Omega} \le c\, h^{-1} \|v_h\|_{0,\Omega} \quad \forall v_h \in \bar{V}_h$

b) $\|\underset{C(\bar\Omega)}{v_h}\|_{\underset{L_\infty(\Omega)}{0,\infty,\Omega}} \le c\, h^{-d/2} \|v_h\|_{\underset{L_2(\Omega)}{0,\Omega}} \quad \forall v_h \in \bar{V}_h$

with generic (h-independent) constants $c > 0$.

Proof:

a) mms, cf. also Section 3.3: $K_h \underline{u}_h = \lambda \underline{u}_h : 0 < \frac{1}{c} \le \lambda \le \bar{c} h^{-2}$!

b) $$\|v_h\|_{L_\infty(\Omega)} \underset{\in C(\bar\Omega)}{=} \max_{x \in \bar\Omega = \cup \bar\delta_r} |v_h(x)| \underset{\exists r \in R_h}{=} \max_{x \in \bar\delta_r} |v_h(x)|$$

$$\underset{\delta_r \to \Delta}{=} \max_{\xi \in \bar\Delta} |v_h(x_{\delta_r}(\xi))|$$

$$= \max_{\xi \in \bar\Delta} \Big| \sum_{\alpha \in A_r = A} v^{(r,\alpha)} p^{(\alpha)}(\xi) \Big|$$

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$$= \max_{\xi \in \hat{\Delta}} \left| \sum_{\alpha \in A_r = A} v^{(r,\alpha)} p^{(\alpha)}(\xi) \right|$$

$\boxed{\text{Equivalence of norms (here: } C(\hat{\Delta})\text{- and } L_2(\Delta)\text{-norms) in the finite-dimensional space } S(\Delta) \text{ with } \dim S(\Delta) = |A|, \text{ i.e. } \exists\, c_A(\Delta) = \text{const} > 0,\ c_A(\Delta) \neq c(h), \text{ cf. also Ex. 3.11.}}$

$$\leq c_A(\Delta) \left\| \sum_{\alpha \in A} v^{(r,\alpha)} p^{(\alpha)}(\xi) \right\|_{L_2(\Delta)}$$

$$= c_A(\Delta) \, \| v_h(x_{\delta_r}(\xi)) \|_{0,\Delta}$$

$$\overset{\Delta \to \delta_r}{=} c_A(\Delta) \sqrt{\int_{\delta_r} (v_h(x))^2 \underbrace{|J_{\delta_r}^{-1}|}_{= 1/|J_{\delta_r}|} \, dx}$$

$$\leq c_A(\Delta) \, \underline{c}_1^{-0.5} h^{-\frac{d}{2}} \| v_h \|_{0,\delta_r}$$

$$\leq \underbrace{c_A(\Delta) \, \underline{c}_1^{-0.5}}_{= c} h^{-\frac{d}{2}} \| v_h \|_{0,\Omega} .$$

q.e.d.

Ex. 3.11 Show that, for linear triangular element $\triangle$ ($d=2$, $K=1$, $S(\Delta) = P_1$), we have

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$$c_A(\Delta) = (\lambda_{min}(G_0))^{-1/2}, \quad G_0 = \left[\int_{\Delta} p^{(\alpha)} p^{(\beta)} d\xi \right]_{\alpha,\beta \in A} !$$

Remark 3.12: about inverse inequalities

Let $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ be two norms in V_h. Due to the equivalence of norms in finite-dimensional spaces, there exists $\underline{c} = \underline{c}(h)$ and $\overline{c} = \overline{c}(h) = \text{const} > 0$:

(30) $\underline{c}(h)\, \|v_h\|_{(2)} \leq \|v_h\|_{(1)} \leq \overline{c}(h)\, \|v_h\|_{(2)} \quad \forall v_h \in V_h.$

Questions: $\underline{c} = \underline{c}(h) \gtrsim \underline{\alpha}\, h^{?}$ with $\underline{\alpha} \neq \underline{\alpha}(h)$

$\overline{c} = \overline{c}(h) \lesssim \overline{\alpha}\, h^{?}$ with $\overline{\alpha} \neq \overline{\alpha}(h)$

Technique:
1. map $\delta_r \longrightarrow \Delta$,
2. norm equivalence on $S(\Delta)$ with h-independent, but k-dependent (polynomial degree) constants;
3. return map $\Delta \longrightarrow \delta_r$.

k-Question: k-dependence: $\alpha = \alpha(k)$, where $P_k \subset S(\Delta)$, but $P_{k+1} \not\subset S(\Delta)$?

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Show the inverse inequality

$$\|v_h\|_{L_\infty(\Omega)} \leq c\, h^{-\frac{d}{p}}\, \|v_h\|_{L_p(\Omega)} \quad \forall v_h \in V_h$$

is valid under the assumption of Lemma 3.11!

Theorem 3.13: (L_∞-Convergence)

Ass.: Let the assumptions of Theorem 3.10 (L_2-convergence) be fulfilled.

St.: Then there exists $c = const > 0$, $c \neq c(u,h)$:

$$(28)\quad \|\underbrace{u - u_h}_{\in C(\bar\Omega)}\|_{L_\infty(\Omega)} = \max_{x\in\bar\Omega} |u(x) - u_h(x)| \leq c\, h^{k+1-\frac{d}{2}} |u|_{k+1,\Omega}$$

Proof:

- $u \in W_2^{k+1}(\Omega) \hookrightarrow C(\bar\Omega)$ for $k \geq 1$ and $d = 1,2,3$.

  Taking the interpolant

  $$v_h = int_{V_h}(u) := \sum_{i\in\bar\omega_h} u(x^{(i)})\, p^{(i)}(x) \in \hat V_{gh} \subset \hat V_g$$

  of the solution $u \in \hat V_g \cap W_2^{k+1}(\Omega) \hookrightarrow C(\bar\Omega)$ of (1), and using the triangle inequality, we ob- i

$$(31)\quad \|u - u_h\|_{L_\infty(\Omega)} \leq \underbrace{\|u - v_h\|_{L_\infty(\Omega)}}_{\substack{\text{interpolation error}\\ \text{in the } L_\infty\text{-norm}\\ (32)\ (\downarrow)}} + \underbrace{\|v_h - u_h\|_{L_\infty(\Omega)}}_{\substack{\in \hat V_{0h} \subset \hat V_h,\ \text{i.e.}\\ \text{Lemma 3.11}\\ \text{is applicable.}\\ (29)(\downarrow)\ (\uparrow)}}$$

?

- We first estimate the interpolation error $\|u - \mathrm{int}_{\tilde V_h}(u)\|_{L_\infty(\Omega)}$ in the L_∞-norm by means of Bramble-Hilbert's Lemma:

$$(32)\quad \|u - \mathrm{int}_{\tilde V_h}(u)\|_{L_\infty(\Omega)} \le c\, h^{k+1-\frac{d}{2}} |u|_{H^{k+1}(\Omega)} .$$

Using the mapping principle and Bramble-Hilbert's Lemma 2.17, we obtain

$$(32)\quad \|\underbrace{u - \mathrm{int}_{\tilde V_h}(u)}_{\in C^1(\bar\Omega)}\|_{L_\infty(\Omega)} = \max_{x\in\delta_r} |u(x) - \mathrm{int}_{\tilde V_h}(x)| \qquad \exists\, r = r_u \in R_h$$

$$\overset{\delta_r \to \Delta}{=} \max_{\xi\in\bar\Delta} \Big| \overset{u(x_{\delta_r}(\xi))}{\overset{\shortparallel}{u_r(\xi)}} - \underbrace{\mathrm{int}_{S(\Delta)}(u_r(\xi))}_{= \sum_{\alpha\in A_r}' u_r(\xi^{(\alpha)})\, p^{(\alpha)}(\xi)} \Big|$$

$\exists\, \bar\xi_r = \bar\xi(u_r)$

$$= |\underbrace{u_r(\bar\xi_r) - \mathrm{int}_{S(\Delta)}(u_r(\bar\xi_r))}_{=:\, \ell_{u_r}(u_r(\bar\xi_r))}| = |\ell_{u_r}(u_r(\bar\xi_r))|$$

$\boxed{\text{B \& H - Lemma}}$ Define $\ell_{u_r}(v) := v(\bar\xi_r) - \mathrm{int}_{S(\Delta)}(v(\bar\xi_r))$:

1) Linear ✓
2) $\ell_{u_r} \in [W_2^{k+1}(\Delta)]^*$ (emb.)
3) $\ell_{u_r}(q) = q(\bar\xi_r) - \mathrm{int}_{S(\Delta)}(q(\bar\xi_r)) = 0 \;\; \forall q \in P_k \subset S(\Delta)$

$$\le c_B |u|_{k+1,\Delta} = c_B \left(|u(x_{\delta_r}(\xi))|^2_{k+1,\Delta} \right)^{1/2}$$

$$\underset{\Delta\to\delta_r}{\overset{(8),(10)}{\le}} c_B \left(\bar c\, \frac{h_r^{2(k+1)}}{h_r^{d}} |u(x)|^2_{k+1,\delta_r} \right)^{1/2} \underset{h_r \le h}{\le} c\, h^{k+1-\frac{d}{2}} |u|_{k+1,\Omega}. \quad \blacksquare$$

see Th. 3.6.

- Inequalities (31), (32), (29) b), and Theorems 3.6 and 3.7 yield:

$$\in \bar{V}_{0h} \subset \bar{V}_h$$

(31) $$\|u-u_h\|_{L_\infty(\Omega)} \leq \underbrace{\|u-\mathrm{int}_{\bar{V}_h}(u)\|_{L_\infty(\Omega)}}_{(32)\text{ interp. error}} + \underbrace{\|\mathrm{int}_{\bar{V}_h}(u)-u_h\|_{L_\infty(\Omega)}}_{(29)\text{ b) inverse ineq.}}$$

$$\leq c\,h^{k+1-\frac{d}{2}}|u|_{k+1,\Omega} + c\,h^{-\frac{d}{2}}\|\mathrm{int}_{\bar{V}_h}(u)-u_h\|_{0,\Omega}$$

$$\leq c\,h^{k+1-\frac{d}{2}}|u|_{k+1,\Omega} + c\,h^{-\frac{d}{2}}\Big\{\underbrace{\|\mathrm{int}_{\bar{V}_h}(u)-u\|_{0,\Omega}}_{\text{Proof of Th. 3.6}} + \underbrace{\|u-u_h\|_{0,\Omega}}_{\text{Th. 3.10}}\Big\}$$

$$\leq c\,h^{k+1-\frac{d}{2}}|u|_{k+1,\Omega} + c\,h^{-\frac{d}{2}}\Big\{c_{0,k+1}h^{k+1}|u|_{k+1,\Omega} + c_{B,k+1}h^{k+1}|u|_{k+1,\Omega}\Big\}$$

$$= c\,h^{k+1-\frac{d}{2}}|u|_{H^{k+1}(\Omega)},$$

with different (generic) positive constants $c \neq c(h,u)$.

q.e.d.

Remark 3.14:

1. If the function u fulfils the stronger regularity assumption

$$u \in V_g \cap W_\infty^{k+1}(\Omega),$$

then the interpolation ($\Rightarrow$ approximation) error estimates

$$|u - \mathrm{int}_{V_h}(u)|_{W_\infty^s(\Omega)} \le c\, h^{k+1-s}\, |u|_{W_\infty^{k+1}(\Omega)}$$

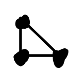
are valid, where $s = 0, 1$ $\quad (W_\infty^0(\Omega) = L_\infty(\Omega))$.

<u>Proof:</u> In analogy to (32) with $\ell \in [W_p^{k+1}(\Omega)]^*$ and then $p \to \infty$!

2. <u>Question:</u> $\|u - u_h\|_{W_\infty^s(\Omega)} \le \; ?\,, \quad s = 0, 1,$

provided that $u \in V_g \cap W_\infty^{k+1}(\Omega)$?

<u>Answer:</u> non-trivial!

(a) For $d = 2$, $k = 1$, $S(\Delta) = P_1$: $\Delta \approx$ ,

i.e. Linear triangular element = Courant's element:

<u>Lit.:</u> e.g. [13] Ciarlet

$$\|u - u_h\|_{L_\infty(\Omega)} \le c\, h^2\, |\log h|^{3/2}\, |u|_{W_\infty^2(\Omega)},$$

$$\|u - u_h\|_{W_\infty^1(\Omega)} \le c\, h\, |\log h|\, |u|_{W_\infty^2(\Omega)}.$$

Techniques for proving such estimates:

① Method of weighted Sobolev spaces proposed by NITSCHE (1975),

② Method of discrete Green's functions proposed by SCOTT [1975].

(b) For "all" (?) other cases, one can prove quasiopt. L_∞- resp. W_∞^1- estimates ($\cong$ approximation)