3.4.3. Convergence in the H^1-norm

■ **Theorem 3.8:** (W_2^1 – Convergence)

Ass.: 1. Standard assumption for variational problems:

1) $F \in V_0^*$ $\quad (V_0 \subset V = W_2^1(\Omega) = H^1(\Omega))$

2) $a(\cdot,\cdot) : V \times V \to \mathbb{R}^1$ ~ continuous bilin. f.:

2a) $a(v,v) \geq \mu_1 \|v\|_1^2 \quad \forall v \in V_0$,

2b) $|a(u,v)| \leq \mu_2 \|u\|_1 \|v\|_1 \quad \forall u,v \in V_0$.

2. Ass. 1 and 2 of Th. 2.6 (approx. theorem),

3. $V_{gh} = g_h + V_{0h} \subset V_g$ – finite dim. hyperplane, with $V_{0h} \subset V_0$ – FE subspace, $g_h \in V_g \cap V_h$ given,

4. $u \in V_g : a(u,v) = \langle F, v \rangle \quad \forall v \in V_0 \quad (1)$

$u_h \in V_{gh} : a(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in V_{0h} \quad (1)_h$

5. Regularity result:

a) $u \in V_g \cap W_2^{k+1}(\Omega)$

b) $u \in V_g$ and $u \in W_2^{k+1}(\delta_r) \quad \forall r \in R_h \quad \forall h \in \Theta$

St.: Then we have the following error estimate:

$$(23) \quad \|u-u_h\|_{1,\Omega} \underset{5b)}{\leq} \underbrace{\frac{\mu_2}{\mu_1} \bar{a}_{1,k+1}}_{=: c_{1,k+1}} \Big[\sum_{r \in R_h} h_r^{2k} |u|_{k+1,\delta_r}^2\Big]^{1/2} \underset{5a)}{\leq} c_{1,k+1} h^k |u|_{k+1,\Omega}$$

Proof follows immediately from (15) = CEA and the approximation Theorem 3.6 resp. Remark 3.7.2 (i.e. (18")):

$$\|u-u_h\|_{1,\Omega} \underset{\text{CEA}}{\leq} \frac{\mu_2}{\mu_1} \inf_{v_h \in V_{gh}} \|u-v_h\|_{1,\Omega} \underset{\text{Th. 3.6}}{\leq} c_{1,k+1} [\ldots]^{1/2} \leq c_{1,k+1} h^k |u|_{k+1,\Omega}$$

q.e.d

Remark 3.9:

1. If, instead of (9) $\|J_{\delta_r}\| \le c_2 h_r$, the more general conditions (22) $|\partial^\beta x_{\delta_r,i}(\xi)| \le \bar{c}_2 h_r^{|\beta|}$ $\forall |\beta| \le k+1$ are assumed, then the error estimate (23') holds:

$$(23') \quad \|u-u_h\|_{1,\Omega} \underset{b)}{\le} c_{1,k+1}\Big[\sum_{r\in R_h} h_r^{2k}\|u\|^2_{k+1,\delta_r}\Big]^{1/2} \underset{a)}{\le} c_{1,k+1} h^k \|u\|_{1,\Omega}$$

2. If $u \in V_g \cap W_2^\ell(\Omega)$ with $1 \le \ell \le k+1$ ($\ell \in \mathbb{R}$), then

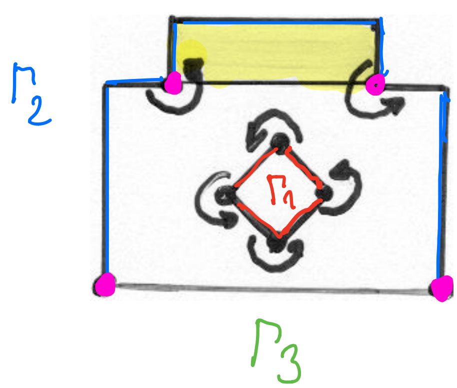
$$\|u-u_h\|_{1,\Omega} \le \frac{\mu_2}{\mu_1}\bar{a}_{1,k+1} h^{\ell-1}\|u\|_{\ell,\Omega}.$$

3. If $u \in W_2^{\ell_r}(\delta_r)$, $1 < \ell_r \le k+1$, $\forall r \in R_h$ $\forall h \in \Theta$, then

$$\|u-u_h\|_{1,\Omega} \le \frac{\mu_2}{\mu_1}\Big[\sum_{r\in R_h}\bar{a}_{1,\ell_r}\underbrace{h_r^{2(\ell_r-1)}}\|u\|^2_{\ell_r,\delta_r}\Big]^{1/2}$$

mesh grading: $\approx h^{2k}$

4. <u>Our Example:</u> $d=2$, $k=1$, $\mathcal{F}(\Delta)=P_1$: $\Delta = $ ◣, $x_{\delta_r}(\cdot) \in P_1$



$u \in W_2^{1+s_r}(\delta_r)$, $0 < s \le s_r \le 1$

$s_r = s_r(\partial\Omega, BC, \dots) = s_r(\bullet)$

$s = s(\partial\Omega, BC, \dots) = s(\bullet)$

$$\Rightarrow \|u-u_h\|_{1,\Omega} \le \frac{\mu_2}{\mu_1}\Big[\sum_{r\in R_h}\bar{a}_{1,1+s_r} h_r^{2s_r}|u|^2_{1+s_r,\delta_r}\Big]^{1/2}$$

$$\le \frac{\mu_2}{\mu_1}\bar{a}_{1,1+s} h^s \Big[\sum_{r\in R_h}|u|^2_{1+s_r,\delta_r}\Big]^{1/2}$$

5. A-priori estimate of $|u|_{k+1,\Omega}$ resp. $\|u\|_{k+1,\Omega}$ by the input data ($= W_2^{k+1}$-coercitivity): e.g. Dirichlet problem for the Poisson equation in a convex domain $\Omega \subset \mathbb{R}^2$ *:

<u>CF</u>: $\quad -\Delta u = f$ in Ω

$\qquad u = 0$ on $\Gamma = \partial\Omega$

<u>VF</u>: $\quad$ Find $u \in V_g = V_0 = \overset{\circ}{W}{}_2^1(\Omega)$:

$$\int_\Omega \nabla^T u \, \nabla v \, dx = \int_\Omega f \cdot v \, dx \quad \forall v \in V_0,$$

with given $f \in L_2(\Omega)$

<u>H^2-coercitivity</u>:

It follows from $f \in L_2(\Omega)$, $\Omega \subset \mathbb{R}^2$ - bounded, Lip, convex:

$$\boxed{\begin{array}{l} 1.\ \exists!\ u \in V_0 \cap H^2(\Omega): \\ 2.\ |u|_{2,\Omega} \le 1 \cdot \|f\|_{0,\Omega} \end{array}}$$

$c = 1$

<u>Proof</u> mms*

$$\int_\Omega f^2 dx = \int_\Omega (-\Delta u)(-\Delta u) dx \overset{\text{integr. by parts}}{=} \dots$$

3.4.4. The NITSCHE-AUBIN Duality Trick and the L_2-Convergence

- **Trivial:** $\|u-u_h\|_{0,\Omega} \le \|u-u_h\|_{1,\Omega} \le c_{1,k+1}\, h^k\, |u|_{k+1,\Omega}$ h?

Approximation: $\inf\limits_{v_h \in \bar{V}_h} \|u-v_h\|_{0,\Omega} \le a_{0,k+1}\, h^{k+1}\, |u|_{k+1,\Omega}$

There is no L_2-Cea-Lemma: $\|u-u_h\|_{0,\Omega} \overset{?}{\le} \inf\limits_{v_h \in \bar{V}_{gh}} \|u-v_h\|_{0,\Omega}$

- **Theorem 3.10:** (L_2-error estimate)

Ass.: 1. Ass. 1.-4. and 5a) from Theorem 3.8 (H^1-Conv.).

2. H^2-coercitivity of the adjoint VP, i.e.

Find $w \in \bar{V}_0 : a(v,w) = \int_\Omega e\,v\,dx \quad \forall v \in \bar{V}_0$,

with some $e \in L_2(\Omega)$, we assume that

1) $\exists!\, w \in \bar{V}_0$ (OK: Lax-Milgram) $: w \in H^{2}(\Omega)$, $H^{1+\alpha}$, $\alpha \in (0,1]$

2) $\exists\, c_c = const > 0$: $2 \mapsto 1+\alpha$

$|w|_{2,\Omega} \le c_c \|e\|_{0,\Omega}$ triv.

resp. $\|w\|_{2,\Omega} \le \tilde{c}_c \|e\|_{0,\Omega}$ (cm ms)

$1+\alpha$

St.: Then there exists $c_{0,k+1} = const > 0$, $c_{0,k+1} \ne c(h)$:

(24) $$\|u-u_h\|_{0,\Omega} \le c_{0,k+1}\, h^{k+1}\, |u|_{k+1,\Omega}$$

Proof: (Aubin-Nitsche-Trick, 1968)

- (1) $u \in V_g : a(u,v) = \langle F, v\rangle \quad \forall v \in V_0$

$-$ (1)$_h$ $u_h \in \bar{V}_{gh} : a(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in \bar{V}_{0h}$

GO (25) Galerkin-orth.: $a(u-u_h, v_h) = 0 \quad \forall v_h \in V_{0h}$

- Consider now the auxiliary (adjoint) problem

TRICK **(26)** Find $w \in \overline{V}_0 : \; a(v,w) = \int_\Omega e_h v \, dx \quad \forall v \in \overline{V}_0$

discretization error $\to e_h = \overset{(1)}{u} - \overset{(1)_h}{u_h} \in \overline{V}_0 \subset L_2(\Omega)$

Due to Ass. 2 of Theorem 3.10, we have

(27)
1) $\exists !\, w \in \overline{V}_0 \cap H^2(\Omega) : (26)$
2) $|w|_{2,\Omega} \le c_c \|e_h\|_{0,\Omega} = c_c \|u - u_h\|_{0,\Omega}$

- Choosing $v = e_h = \underset{V_g}{u} - \underset{V_{gh}}{u_h} \in \overline{V}_0$ in (26), ($V_g \supset V_{gh}$)

we immediately get

$$\|u-u_h\|_{0,\Omega}^2 = \|e_h\|_{0,\Omega}^2 = \int_\Omega e_h \cdot e_h \, dx \overset{(26)}{=} a(e_h, w) =$$

$$= a(u-u_h, w) = a(u-u_h, w - w_h)$$

GO (25) $a(u-u_h, w_h) = 0 \quad \forall w_h \in \overline{V}_{0h}$

$w_h = \mathrm{int}_{V_h}(w) \in \overline{V}_{0h}$! $\quad w \in C(\overline{\Omega})$ (embedding)

$$\le \mu_2 \|u-u_h\|_{1,\Omega} \, \|w - w_h\|_{1,\Omega} \le$$

Theorem 3.8: H^1 - discretization error estimate

Theorem 3.6: approximation theorem, Remark 3.7

$$\le \mu_2 c_{1,k+1} h^k |u|_{k+1,\Omega} \cdot \bar{a}_{1,2} h |w|_{2,\Omega}$$

$$\overset{(27)}{\le} \mu_2 c_{1,k+1} \bar{a}_{1,2} h^{k+1} |u|_{k+1,\Omega} \cdot c_c \|u-u_h\|_{0,\Omega}$$

h^{k+s}

$$\Longrightarrow \|u-u_h\|_{0,\Omega} \le \underbrace{\mu_2 c_{1,k+1} \bar{a}_{1,2} c_K}_{=: c_{0,k+1}} h^{k+1} |u|_{k+1,\Omega}.$$

q.e.d.