- It remains to prove estimate

$$\boxed{\underbrace{|u(x_{\delta_r}(\xi)) - \mathrm{int}_{\tilde V_h}(u)(x_{\delta_r}(\xi))}_{= e_h(x_{\delta_r}(\xi))}|_{1,\Delta} \le c_B\, |u(x_{\delta_r}(\xi))|_{2,\Delta}} \quad (20)$$

of the interpolation error on the master element Δ.

To prove (20), we need the Bramble-Hilbert Lemma 2.17:

<u>Ass.:</u> $\Delta \subset \mathbb{R}^d$ ⚡ and Lip, $1 \le p < \infty$, $K \in \{0,1,\dots\}$,

$\ell(\cdot) := \langle \ell, \cdot \rangle \in [W_p^{K+1}(\Delta)]^*$: $\ell(q) = 0 \;\forall q \in \mathcal{P}_K$,

<u>St.:</u> $|\ell(u)| \le c\, |u|_{K+1,p,\Delta} \quad \forall u \in W_p^{K+1}(\Delta)$,

with $c = C(\Delta)\, \|\ell\|_*$, $C(\Delta) = \mathrm{const}(\Delta, p, K)$,

$\|\ell\|_* \ge \|\ell\|_{[W_p^{K+1}(\Delta)]^*}$.

for $d=2$, $p=2$, $K=1$:

<u>Problem:</u> $|e_h|_{1,\Delta}$ is NOT a Linear functional ! ☹

<u>Trick:</u> $$|e_h|_{1,\Delta} = \sup_{w \perp \mathrm{Ker}|\cdot|} \frac{(\nabla e_h, \nabla w)_{0,\Delta}}{|w|_{1,\Delta}} \le ?$$

i.e. $(\nabla e_h, \nabla w)_{0,\Delta} \le ?\; |w|_{1,\Delta}$

Let $w \in H^1(\Delta) = W_2^1(\Delta)$ be an arbitrary, but fixed function, and let us consider the Linear functional

$$\ell(u) := \int_\Delta \nabla_\xi w(\xi)\cdot \nabla_\xi \Big(\overbrace{\underbrace{u(x_{\delta_r}(\xi))}_{=:\, u_r(\xi)} - \underbrace{int_{V_h}\big(u(x_{\delta_r}(\xi))\big)}_{=:\, int_{S(\Delta)}(u_r(\xi))}}^{=\, e_h(x_{\delta_r}(\xi))} \Big)\, d\xi$$

$$\ell(u) = \ell(u)$$

(mms) omitted

$\Longrightarrow$ 1) $\ell(\cdot)$ is Linear, since the interpolation operator is Linear!

2) $\ell(\cdot)$ is bounded (continuous) on $W_2^2(\Delta)$!

Indeed:

$$|\ell(u)| = |\ell(u)| \le |w|_{1,\Delta}\, |u - int_{S(\Delta)}(u)|_{1,\Delta}$$

$$int_{S(\Delta)}(u) = {\sum_{\alpha\in A}}' u(\xi^{(\alpha)})\, p^{(\alpha)}(\xi), \qquad u(\xi^{(\alpha)}) = u(x_{\delta_r}(\xi^{(\alpha)}))$$

triangle inequality

$$\le |w|_{1,\Delta} \Big\{ |u|_{1,\Delta} + \Big| {\sum_{\alpha\in A}}' u(\xi^{(\alpha)})\, p^{(\alpha)}(\xi) \Big|_{1,\Delta} \Big\}$$

$$\le |w|_{1,\Delta} \Big\{ \underbrace{|u|_{1,\Delta}}_{\le \|u\|_{2,\Delta}} + {\sum_{\alpha\in A}}' \underbrace{|u(\xi^{(\alpha)})|}_{\le c_E(\Delta)\, \|u\|_{2,\Delta}}\, |p^{(\alpha)}|_{1,\Delta} \Big\}$$

$(19)_\Delta$ $\boxed{(19)_{\Omega=\Delta} \text{ Embedding: } W_2^2(\Delta) \hookrightarrow C(\bar\Delta)}$

$$\le \underbrace{|w|_{1,\Delta} \Big\{ 1 + c_E(\Delta) {\sum_{\alpha\in A}}' |p^{(\alpha)}|_{1,\Delta} \Big\}}_{=:\, \|\ell\|_*} \|u\|_{2,\Delta}\,.$$

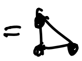
L16-03

3) $\ell(q) = 0 \ \forall q \in P_1 \qquad (P_K)$

since $q(\xi) = \mathrm{int}_{S(\Delta)}(q(\xi)) \ \forall \xi \in \bar{\Delta}$

$\uparrow$

$P_1 \subset S(\Delta)$, e.g. $S(\Delta) = P_1$

for $\Delta =$ 

$\Longrightarrow$ B & H - Lemma 2.17 gives:

$$|\ell(u)| \lesssim c(\Delta) \, \|\ell\|_* \, |u|_{2,\Delta}$$

$$= c(\Delta) \underbrace{\Big\{1 + c_E(\Delta) \sum_{\alpha \in A} |p^{(\alpha)}|_{1,\Delta}\Big\} |w|_{1,\Delta}}_{= \|\ell\|_*} \, |u(x_{\delta_r}(\xi))|_{2,\Delta}$$

$\Longrightarrow$ Now we choose

$$w = e_h = u - \mathrm{int}_{S(\Delta)}(u)$$

$$= u(x_{\delta_r}(\xi)) - \mathrm{int}_{S(\Delta)}\big(u(x_{\delta_r}(\xi))\big) \in W_2^1(\Delta)$$

which gives us the estimate

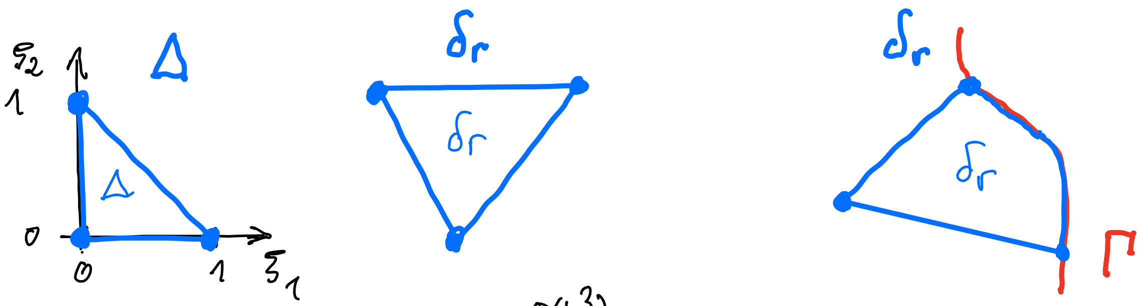
$$|e_h|^2_{1,\Delta} = \ell(u) \leq$$

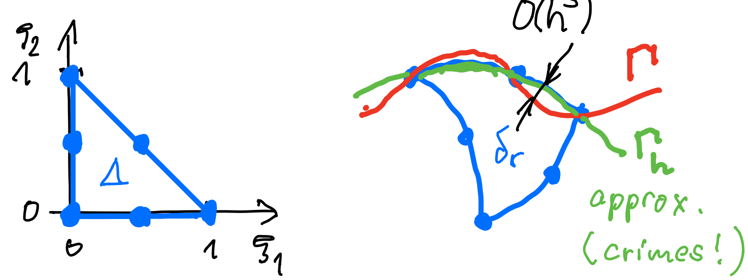
$$\lesssim \underbrace{c(\Delta)\Big\{1 + c_E(\Delta) \sum_{\alpha \in A} |p^{(\alpha)}|_{1,\Delta}\Big\}}_{=: c_B} |e_h|_{1,\Delta} \, |u(x_{\delta_r}(\xi))|_{2,\Delta} \qquad (20)$$

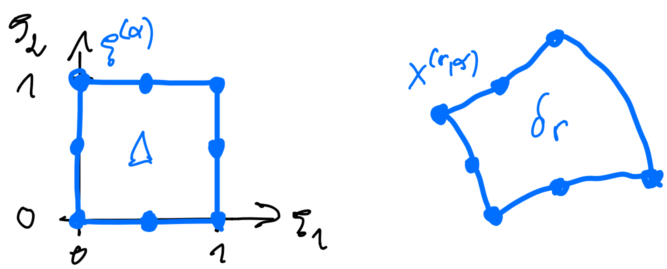
q.e.d.

Remark 3.7:

1. Falls $x_{\delta_r}(\cdot) \in S(\Delta) \supset P_k$ oder $x_{\delta_r}(\cdot) \in C^{k+1}(\bar{\Delta})$







$$x_{\delta_r}(\xi) = \sum_{\alpha \in A}{}' \, x^{(r,\alpha)} \, p^{(\alpha)}(\xi)$$

isoparametric map	general nonlinear map

(22) $$\left| \frac{\partial^{|\beta|} x_{\delta_r,i}(\xi)}{\partial \xi^\beta} \right| \le \bar{c}_2 h^{|\beta|} \quad \forall \beta : |\beta| \le k+1, \ \forall \xi \in \bar{\Delta}, \ \forall i = \overline{1,d}, \ \forall r \in R_h, \ \forall h \in \Theta,$$

then we have (cf. **(21)** in proof step 4!) $\qquad m = k+1-s$

(18') $$\inf_{v_h \in \tilde{V}_h} |u - v_h|_{s,\Omega} \le \tilde{a}_{s,k+1} \Big(\sum_{r \in R_h}{}' h_r^{2m} \|u\|^2_{k+1,\delta_r} \Big)^{1/2} \le a_{s,k+1} h^m \|u\|_{k+1,\Omega}$$

2. Estimates (18) or (18') immediately yield:

(18'') $$\inf_{v_h \in \bar{V}_h} \| u - v_h \|_{1,\Omega} \le \bar{a}_{1,k+1} h^k \begin{cases} |u|_{k+1,\Omega}, & \text{Th. 3.6} \\ \|u\|_{k+1,\Omega}, & \text{Rem. 3.7.1.} \end{cases}$$

with $\bar{a}^2_{1,k+1} = a^2_{1,k+1} + a^2_{0,k+1} h^2$.

3. Estimates (18), (18'), (18'') are optimal wrt the h-power (cf. Ex. 3.9), i.e.

$$\exists u \in W_2^{k+1}(\Omega): \quad \inf_{v_h \in V_h} |u - v_h|_{s,\Omega} \ge c h^{k+1-s}$$

with some h-independent positive constant c.

4. For $u \in W_2^{\ell}(\Omega)$, $1 < \ell \le k+1$ ($\ell \in \mathbb{R}$, real!), the following estimates hold:

$$\inf_{v_h \in \bar{V}_h} |u - v_h|_{s,\Omega} \le a_{s,\ell} h^{\ell - s} \|u\|_{\ell,\Omega}.$$

$s \in [0,1]$

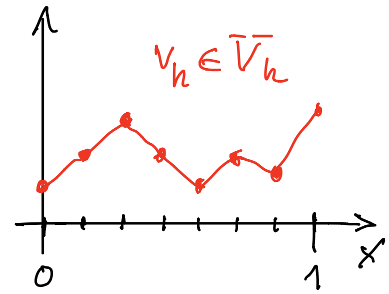
5. If $u \in W_2^{\ell_r}(\delta_r)$, $1 < \ell_r \le k+1$, $\forall r \in R_h$, $\forall h \in \oplus$, then we have the estimate

$$\inf_{v_h \in \bar{V}_h} |u - v_h|_{s,\Omega} \le \Big[\underbrace{\sum_{r \in R_h} a_{s,\ell_r} h_r^{2(\ell_r - s)} \|u\|^2_{\ell_r, \mathrm{int}(\delta_r)}}_{\text{can be used for a priori mesh grading!}} \Big]^{1/2}$$

δ_r in 2d

$\boxed{\text{Ex. 3.9.}}$ Show that, for $d=1$, $\Omega=(0,1)$,
$k=1$: $S(\Delta)=P_1$, $u(x)=x^2$,
we have

$$\inf_{v_h\in\bar{V}_h} \int_0^1 |u'-v_h'|^2 dx = \frac{1}{3}h^2$$



$\boxed{\text{Ex. 3.10}}$ Show the completness of the family of the FE-spaces $\{\bar{V}_{0h}\}_{h\in\Theta}$ in $\bar{V}_0$, i.e.

$$\lim_{\substack{h\to 0\\ h\in\Theta}} \inf_{v_h\in\bar{V}_0} \|u-v_h\|_{\bar{V}_0} = 0$$

by means of the approximation Theorem 3.6, e.g., for $V_0=\overset{\circ}{H}{}^1(\Omega)=\overline{\overset{\circ}{C}{}^\infty(\Omega)}^{\|\cdot\|_{1,\Omega}}$!