3.4. A priori Discretization Error Estimates

3.4.1. Road Map for the Error Analysis

- **Starting Point**

= **Cea's theorem** (lemma) [see NuPDE]

with $V = H^1(\Omega) = W_2^1(\Omega)$, $\|\cdot\| = \|\cdot\|_1 = \|\cdot\|_{H^1(\Omega)}$, i.e. for scalar 2nd order PDEs:

$$(15) \qquad \underbrace{\| \underset{(1)}{u} - \underset{(1)_h}{u_h} \|_1}_{\text{discretization error}} \le \frac{\mu_2}{\mu_1} \underbrace{\inf_{v_h \in \bar{V}_{gh}} \| \underset{(1)}{u} - v_h \|_1}_{\text{approximation error}}$$

Proof is based on the socalled Galerkin orthogonality:

$$(1) \qquad u \in \bar{V}_g : a(u, w_h) = \langle F, w_h \rangle \quad \forall w_h \in V_{0h} \subset \bar{V}_0$$

$$- \qquad u_h \in \bar{V}_{gh} : a(u_h, w_h) = \langle F, w_h \rangle \quad \forall w_h \in \bar{V}_{0h}$$

$$\text{GO} \qquad a(u - u_h, w_h) = 0 \quad \forall w_h \in \bar{V}_{0h}$$

Choose $w_h = u - u_h - (u - v_h) = v_h - u_h \in \bar{V}_{0h} \quad \forall v_h \in \bar{V}_{gh}$

$$\mu_1 \|u - u_h\|_1^2 \le a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \le \mu_2 \|u - u_h\|_1 \|u - v_h\|_1$$

$$\Rightarrow \qquad \|u - u_h\|_1 \le \frac{\mu_2}{\mu_1} \|u - v_h\|_1 \quad \forall v_h \in \bar{V}_{gh}. \qquad \textbf{q.e.d.}$$

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Road Map for the Proof:

- Choose

$$v_h = \underset{\substack{\uparrow\\ \text{interpolant}}}{int_{\overline{V}_h}}(u) \overset{\uparrow}{=} \sum_{i \in \overline{\omega}_h} u(x^{(i)})\, p^{(i)}(x)$$

$u \in \overline{V}_g \cap C(\overline{\Omega})$, e.g., due to Sobolev's embedding theorem:

$W_p^{k+1}(\Omega) \hookrightarrow C'(\overline{\Omega})$

if $(k+1)\,p > d$,

we have for $d=2$

$(\Omega \subset \mathbb{R}^2 : \not\Upsilon \wedge Lip)$:

$u \in \overline{V}_g \cap H^2 \Rightarrow u \in \overline{V}_g \cap C(\overline{\Omega})$

since $2 \times 2 > d = 2$

$$= \underbrace{\sum_{i \in \omega_h} u(x^{(i)})\, p^{(i)}(x)}_{\substack{= 0 \text{ on } \Gamma_1 \\ \in \overline{V}_{0h} \subset \overline{V}_0}} + \underbrace{\sum_{i \in \gamma_h} \underset{\substack{\| \\ g_1(x^{(i)})}}{u(x^{(i)})}\, p^{(i)}(x)}_{\substack{\overset{\approx}{=} g_1(x)\ \forall x \in \Gamma_1 \\ \text{"crime"} \\ \in \overline{V}_{gh} \subset \overline{V}_g}}$$

or, if $u \notin C'(\overline{\Omega})$, choose Clément interpolator (Subsec. 3.6.1):

$$\Rightarrow \underbrace{\inf_{v_h \in \overline{V}_{gh}} \| u - v_h \|_1}_{\text{approximation error}} \overset{\uparrow}{\leq} \underbrace{\| \overbrace{u - int_{\overline{V}_h}(u)}^{\bullet} \|_1}_{\text{interpolation error}} =: \| \bullet \|_1 \underset{(\downarrow)}{\leq} \overset{?}{\dots} \underset{(h \to 0)}{\leq} ?\, h^{?}\, ?$$

$v_h = int_{\overline{V}_h}(u) \in \overline{V}_{gh}$

- $\overset{\delta_r}{\text{element}}$ — $\overset{x = x_{\delta_r}(\xi)}{\text{map to}} \rightarrow \underset{\text{estimate}}{\boxed{\overset{\Delta}{\text{master element}}}}$ — $\overset{\xi = \xi_{\delta_r}(x)}{\text{map back to}} \rightarrow \overset{\delta_r}{\text{element}}$:

$$\|\bullet\|^2_{1,\Omega} = \sum_r \|\bullet\|^2_{1,\delta_r} = \sum_r \int_{\delta_r} [\ldots] \, dx =$$

$$x = x_{\delta_r}(\xi) \longrightarrow \overset{\downarrow}{=} \sum_r \int_{\Delta} [\,..\, ? \,..\,] \, |J_{\delta_r}| \, d\xi$$

$$\le \sum_r c h^{?} \underbrace{\|\bullet\|^2_{1,\Delta}} \le \ldots \overset{\xi = \xi_{\delta_r}(x)}{\overset{\downarrow}{\lesssim}} \; ? \, h^{?} \, ?$$

estimation of the interpolation error on the master element Δ with the help of the BRAMBLE-HILBERT Lemma 2.17 !

- <u>Result</u> $\Rightarrow$ apriori discretizion error estimate w.r.t. the $\|\cdot\|_1$-norm (H^1-norm), see Subsections 3.4.2 and 3.4.3. !

- **Apriori error estimates with respect to (w.r.t.) other norms are also interesting:**
  - L_2-norm $\|\cdot\|_{L_2(\Omega)} = \|\cdot\|_{0,\Omega}$ → 3.4.4.
  - L_∞-norm $\|\cdot\|_{L_\infty(\Omega)} = \|\cdot\|_{0,\infty,\Omega}$ → 3.4.5.
  - W^1_∞-norm $\|\cdot\|_{W^1_\infty(\Omega)} = \|\cdot\|_{1,\infty,\Omega}$ → 3.4.5.
  - L_p-norm $\|\cdot\|_{L_p(\Omega)} = \|\cdot\|_{0,p,\Omega}$ → Literature
  - W^1_p-norm $\|\cdot\|_{W^1_p(\Omega)} = \|\cdot\|_{1,p,\Omega}$ → Literature

    ⋮
  - Goal-oriented estimates: $\ell \in V_0^*$ or $\ell \in (V_0 \cap H^2)^*$

    $|\ell(u) - \ell(u_h)| = |\ell(u - u_h)| \le \; ? \, h^{?} \, ?$ (apriori !).

    See also Section 3.6: a posteriori error estimates !

3.4.2. The Approximation Theorem

Theorem 3.6: (approximation theorem)

Ass.: 1. Let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be provided with a regular triangulation in the sense of Def. 3.3, i.e.

$$\forall h \in \textcircled{H} : \bar{\Omega} = \bigcup_{r \in R_h} \delta_r \,, \quad \delta_r \underset{\xi = \xi_{\delta_r}(x)}{\overset{x = x_{\delta_r}(\xi)}{\rightleftarrows}} \Delta \quad \forall r \in R_h :$$

$h \rightsquigarrow h_r$

Shape reg.

$$(8) \quad \underline{c}_1 h_r^d \leq |J_{\delta_r}| \leq \bar{c}_1 h_r^d \qquad \forall \xi \in \bar{\Delta} \ \forall r \in R_h ,$$

$$(9) \quad \| J_{\delta_r} \| := (\lambda_{max}(J_{\delta_r}^T J_{\delta_r}))^{1/2} \leq c_2 h_r \qquad -\text{''}- ,$$

$$(10) \quad \| J_{\delta_r}^{-T} \| = \| J_{\delta_r}^{-1} \| \leq c_3 h_r^{-1} \qquad -\text{''}- ,$$

where, for the time being, $x_{\delta_r}(\cdot) \in P_1(\Delta)$, i.e. an affine linear mapping (see Remark 3.7 for generalizations).

$A \rightsquigarrow A_r$
$S \rightsquigarrow S_r$

2. $S(\Delta) = \text{span}\{ p^{(\alpha)} : \alpha \in A \} \supset P_k(\Delta)$,

3. $u \in W_2^{k+1}(\Omega)$, or more general, $u \in W_2^{k+1}(\delta_r) \ \forall r \in R_h \ \forall h \in \textcircled{H}$.

a) $\Longrightarrow$ b)

St.: $\exists \ \tilde{a}_{s,k+1} = \text{const} > 0$ (indepentent of h and u):

$$(18) \quad \inf_{v_h \in \bar{V}_h} |u - v_h|_{s,\Omega} \overset{b)}{\leq} \tilde{a}_{s,k+1} \Big[\sum_{r \in R_h} h_r^{2(k+1-s)} |u|_{k+1,\delta_r}^2 \Big]^{1/2} \ b)$$

$W_2^s(\Omega)$

$$\overset{b)}{\leq} a_{s,k+1} \Big[\sum_{r \in R_h} h^{2(k+1-s)} |u|_{k+1,\delta_r}^2 \Big]^{1/2} \ b)$$

$$\overset{a)}{=} a_{s,k+1} h^{k+1-s} |u|_{k+1,\Omega} , \qquad a)$$

where $s = 0,1$, or $s \in [0,1]$ (interpolation theory).

Proof: Without loss of generality (WLG), we prove the theorem for the case:

$d=2$, $K=1$, $\Delta =$ , i.e. $S(\Delta)=\mathcal{P}_1$, $s=1$ ($s=0$: mms), i.e.

$$(18)_{1,2} \qquad \inf_{v_h \in \overline{V}_h} |u-v_h|_{1,\Omega} \leq a_{1,2}\, h\, |u|_{2,\Omega}. \qquad a)$$

- Thus, Let $u \in W_2^2(\Omega) = H^2(\Omega)$ and $\Omega \subset \mathbb{R}^2 *$, $\partial\Omega \in C^{0,1}$. It follows from Sobolev's embedding Theorem 2.26 that
$$W_p^{k+1}(\Omega) \hookrightarrow C(\overline{\Omega}) \text{ if } (k+1)p > d,$$
i.e., for $p=2$, we always have
$$(k+1)\cdot 2 \geq 4 > d = 1,2,3 \text{ for } k=1,2,\ldots .$$
Therefore: $W_2^2(\Omega) = H^2(\Omega) \hookrightarrow C(\overline{\Omega})$, i.e., in particular, $\exists\, c_E = c_E(\Omega) = \text{const} > 0$:

$$(19)_\Omega \qquad \boxed{\max_{x\in\overline{\Omega}} |u(x)| \leq c_E \|u\|_{2,\Omega} \quad \forall u \in W_2^2(\Omega)}$$

- <u>Now the proof consists of 4 steps:</u>

1. Insert the <u>interpolant</u>
$$v_h(x) = \operatorname{int}_{\overline{V}_h}(u)(x) := \sum_{i\in\overline{\omega}_h} u(x^{(i)})\, p^{(i)}(x) \in \overline{V}_{gh}$$
into the infimum (18):
$$\overset{s=1}{\Longrightarrow} \quad \underbrace{\inf_{v_h\in\overline{V}_h} |u-v_h|_{1,\Omega}}_{\text{approximation error}} \leq \underbrace{|u-\operatorname{int}_{\overline{V}_h}(u)|_{1,\Omega}}_{\text{interpolation error}}$$

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2. Mapping: $\delta_r \longrightarrow \Delta$

$$\underbrace{|u-\mathrm{int}_{V_h}(u)|}_{=:\, e_h}{}^2_{1,\Omega} = \sum_{r\in\mathbb{R}_h} \int_{\delta_r} |\nabla_x e_h(x)|^2 dx$$

$\nabla_x = J_{\delta_r}^{-T} \nabla_x$ (see Subsect. 3.2.4)

$$= \sum_{r\in\mathbb{R}_h} \int_{\Delta} | J_{\delta_r}^{-T} \nabla_\xi e_h(x_{\delta_r}(\xi))|^2 |J_{\delta_r}| d\xi$$

$$\boxed{\begin{array}{l} (8)\ \underline{\tilde{c}}_1 h_r^d \leq |J_{\delta_r}| \leq \bar{c}_1 h_r^d \\ (10)\ \|J_{\delta_r}^{-T}\| \leq \tilde{c}_3 h_r^{-1} \\ \alpha_0 h \leq h_r \leq h \quad \leq \underbrace{\frac{\tilde{c}_3}{\alpha_0}}_{= c_3} h^{-1} \\ h_r^{-1} \leq \frac{1}{\alpha_0} h^{-1} \end{array}}$$

$$\leq \bar{c}_1 \tilde{c}_3^2 \sum_{r\in\mathbb{R}_h} h^{d-2} \underbrace{\int_{\Delta} |\nabla_\xi e_h(x_{\delta_r}(\xi))|^2 d\xi}_{= |e_h(x_{\delta_r}(\xi))|^2_{1,\Delta}} \leq$$

3. Application of Bramble-Hilbert's Lemma 2.17 on the master (= reference) element Δ:

$$(20) \quad \boxed{|e_h(x_{\delta_r}(\xi))|_{1,\Delta} \leq c_B |u(x_{\delta_r}(\xi))|_{\overset{2,\Delta}{k+1,\Delta}}} \quad (\downarrow)$$

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$$\leq \bar{c}_1 \tilde{c}_3^2 c_B^2 \sum_{r\in\mathbb{R}_h} h_r^{d-2} \underbrace{\int_{\Delta} \sum_{\substack{|\alpha|=2 \\ k+1}} |\partial_\xi^\alpha u(x_{\delta_r}(\xi))|^2 d\xi}_{} =$$

$d=2$:

$$= \int_{\Delta} \left\{ \left(\frac{\partial^2 u}{\partial \xi_1^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial \xi_1 \partial \xi_2}\right)^2 + \left(\frac{\partial^2 u}{\partial \xi_2^2}\right)^2 \right\} d\xi_1 d\xi_2$$

4. Return mapping: $\Delta \longrightarrow \delta_r$

$$= \bar{c}_1 \tilde{c}_3^2 c_B^2 \sum_{r \in \mathbb{R}_h} h_r^{d-2} \int_{\delta_r} \sum_{|\alpha| = \underset{K+1}{2}} |\partial_\xi^\alpha u(x)|^2 \underbrace{|J_{\delta_r}^{-1}|}_{= 1/|J_{\delta_r}|} dx$$

$$\nabla_\xi \otimes \nabla_\xi = \begin{pmatrix} \frac{\partial^2}{\partial \xi_1} \\ \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \\ \frac{\partial^2}{\partial \xi_2 \partial \xi_1} \\ \frac{\partial^2}{\partial \xi_2^2} \end{pmatrix}$$

$$\boxed{\begin{aligned} &\sum_{|\alpha| = \underset{K+1}{2}} |\partial_\xi^\alpha u(x)|^2 = |\nabla_\xi \otimes \nabla_\xi u|^2 = |J_{\delta_r}^T \nabla_x \otimes J_{\delta_r}^T \nabla_x u|^2 \\ (21)\quad &\leq \|J_{\delta_r}^T\|^{\overset{2(K+1)}{4}} \|\nabla_x \otimes \nabla_x u\|^2 = \|J_{\delta_r}\|^{\overset{2(K+1)}{4}} \sum_{|\alpha|=2} |\partial_x^\alpha u|^2 \end{aligned}}$$

$$\leq \bar{c}_1 \tilde{c}_3^2 c_B^2 \sum_{r \in \mathbb{R}_h} h_r^{d-2} \underset{\sim}{\tilde{c}}_1^{-1} h_r^{-d} c_2^4 h_r^4 \int_{\delta_r} \sum_{|\alpha|=2} |\partial_x^\alpha u(x)|^2 dx$$

$$= \underbrace{\frac{\bar{c}_1}{\underset{\sim}{\tilde{c}}_1} \tilde{c}_3^2 c_B^2 c_4^4}_{\overset{!!}{=}\ \tilde{a}_{1,2}^2} \sum_{r \in \mathbb{R}_h} h_r^2 \int_{\delta_r} \sum_{|\alpha|=2} |\partial_x^\alpha u|^2 dx$$

$$\overset{a)}{\leq} a_{1,2}^2 h^2 |u|_{2,\Omega}^2 \qquad \text{(q.e.d.)}$$

But it remains to prove estimate (20) $|e_h|_{1,\Delta} \leq c_B |u|_{2,\Delta}$;
see next Lecture 16!