3.3. Properties of the System $K_h \underline{u}_h = \underline{f}_h$ of Finite Element Equations

- **1d:** see Lectures on NuPDEs by Prof. Zulehner
- The system of FE eqns $K_h \underline{u}_h = \underline{f}_h$ resp. the stiffness matrix K_h have the following properties:

1. **Large-scale:** $\Omega \subset \mathbb{R}^d$ (2d: d=2 ; 3d: d=3)

   $N = N_h = |\omega_h| =$ number of unknowns $=$ DOF $= O(h^{-d})$,

   where $h = N^{-1/d}$ – discretization (mesh) parameter:

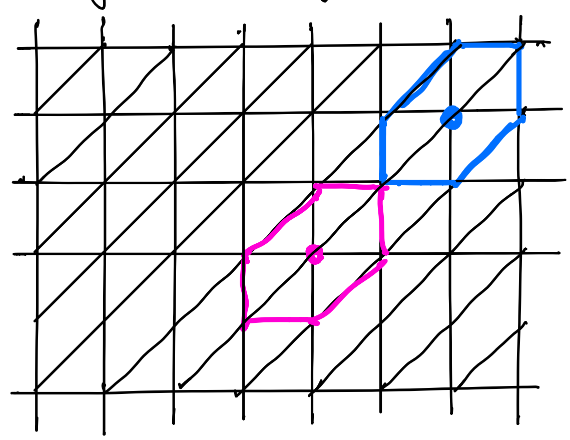
   $\longrightarrow$ in practic: $N = 10^5 - 10^7$ – $\underbrace{10^9 \text{ and beyond}}_{\text{supercomputing}}$

2. **sparse:**

   $K_{ij} = 0 \quad \forall i,j \in \omega_h : \text{supp}(p^{(i)}) \cap \text{supp}(p^{(j)}) = \emptyset$, i.e.

   $B_{ij} := B_i \cap B_j = \emptyset$

   $\bigcup_{r \in B_i} \bar{\delta}_r = \text{supp}(p^{(i)})$



   $NNE(i) =$ Number of Non-zero Elements (NNE) in the i-th row $= |\{ j \in \omega_h : x^{(j)} \in \bigcup_{r \in B_i} \bar{\delta}_r \}| = \underset{\text{reg. mesh}}{O(1)}$

   $NNE = \sum_{i \in \omega_h} NNE(i) = NNE$ of $K_h = O(h^{-d})$

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3. Band-Width (BW) resp. Profile (SL = Sky Line):

→ depend on the global numbering of the nodes!

→ in general: $BW = BW(K_h) = O(h^{-(d-1)})$

$SL = SL(K_h) = O(h^{-2d+1})$

Order estimates cannot be improved!

→ minimization algorithms for BW resp. SL, i.e. find node numbering s.t. BW resp. SL → min = NP-hard!

→ see Literature, e.g. [JL, 2013: pp. 462-476] for heuristic algorithms:

Alg. 5.16: Cuthill-McKee

Alg. 5.17: Reverse Cuthill-McKee

Alg. 5.18: Minimum degree

4. Inheritance Relation:

In general, the properties of the bilinear form $a(\cdot,\cdot)$ carry over to the stiffness matrix K_h: (mms)

(7) $$\boxed{(K_h \underline{u}_h, \underline{v}_h) = a(u_h, v_h) \quad \forall \underline{u}_h, \underline{v}_h \overset{FE}{\underset{iso}{\longleftrightarrow}} u_h, v_h \in V_{0h}}$$

For instance:

- $a(\cdot,\cdot)$ symmetric $\Longrightarrow K_h = K_h^T$, since

$$(K_h \underline{u}_h, \underline{v}_h) \overset{(7)}{=} a(u_h, v_h) = a(v_h, u_h) \overset{(7)}{=} (K_h \underline{v}_h, \underline{u}_h) = (\underline{u}_h, K_h \underline{v}_h) \blacksquare$$

- $a(\cdot,\cdot)$ V_0-positive $\Longrightarrow K_h$ positive definite, since

$$(K_h \underline{u}_h, \underline{u}_h) \overset{(7)}{=} a(u_h, u_h) > 0 \quad \forall \underline{u}_h \Leftrightarrow u_h \in V_{0h} : \underline{u}_h \neq 0 \blacksquare$$

$\longrightarrow K_h$ is SPD !

■ Here and in the following, we suppose that the triangulation is **regular** (see (4) in Subsection 3.2.2). The following definition of a **regular triangulation** is more general than (4):

Def. 3.3: (regular triangulation)
A family $\{\mathcal{T}_h\}_{h \in \Theta}$ of triangulations $\mathcal{T}_h = \{\delta_r : r \in R_h\}$ is called **regular**, if there exists positive, h-independent constants $\underline{c}_1, \bar{c}_1, c_2, c_3 = const > 0$ such that

(8) $\underline{c}_1 h^d \le |J_{\delta_r}| := |\det J_{\delta_r}| \le \bar{c}_1 h^d, \ \forall \xi \in \bar{\Delta} \ \forall r \in R_h \ \forall h \in \Theta,$

(9) $\|J_{\delta_r}\| := \sqrt{\lambda_{max}(J_{\delta_r}^T J_{\delta_r})} \le c_2 h, \ \forall \xi \in \bar{\Delta} \ \forall r \in R_h \ \forall h \in \Theta,$

(10) $\|J_{\delta_r}^{-T}\| := \sqrt{\lambda_{max}(J_{\delta_r}^{-1} J_{\delta_r}^{-T})} \le c_3 h^{-1}, \ \forall x \in \bar{\delta}_r \ \forall r \in R_h \ \forall h \in \Theta,$

where $J_{\delta_r} = \dfrac{\partial x_{\delta_r}}{\partial \xi}$ and $J_{\delta_r}^{-1} = \dfrac{\partial \xi_{\delta_r}}{\partial x}$, $\delta_r \underset{x = x_{\delta_r}(\xi)}{\overset{\xi = \xi_{\delta_r}(x)}{\rightleftarrows}} \Delta$.

Theorem 3.4: (eigenvalue and condition number estimates)

Ass.: 1. $a(\cdot,\cdot) : V \times V \to \mathbb{R}^1$ - bilinear form, $V = H^1(\Omega)$, $\|\cdot\| = \|\cdot\|_1$:

a) V_0-elliptic: $a(v,v) \geq \mu_1 \|v\|_1^2 \quad \forall v \in V_0$,

b) V_0-bounded: $|a(u,v)| \leq \mu_2 \|u\|_1 \|v\|_1 \quad \forall u,v \in V_0$

c) V_0-symmetric: $a(u,v) = a(v,u) \quad \forall u,v \in V_0$

2. Regular triangulation $\{\mathcal{T}_h\}_{h \in \Theta}$ of $\Omega \subset \mathbb{R}^d$ in the sense of Def. 3.3: (8)-(10).

St.: Then the following estimates are valid:

1. $\exists\, \underline{c}_E, \overline{c}_E = \mathrm{const} > 0 : c_E \neq c_E(h)$ and

(11) $$\underline{\gamma} = \underline{c}_E h^d \leq \overset{SPD}{\lambda(K_h)} \leq \overline{c}_E h^{d-2} = \overline{\gamma},$$

eigenvalues of K_h

2. $$\varkappa(K_h) = \mathrm{cond}_2(K_h) := \|K_h\|_2 \|K_h^{-1}\|_2 \overset{SPD}{=} \frac{\lambda_{max}(K_h)}{\lambda_{min}(K_h)} \leq \frac{\overline{c}_E}{\underline{c}_E} h^{-2}.$$

spectral condition number

Proof:

- Starting point for the eigenvalue (EV) estimates = Rayleigh-quotient representation:

(11) $$\underline{\gamma} \overset{!}{\leq} \lambda_{min}(K_h) = \min_{\substack{u_h \in \mathbb{R}^{N_h} \\ u_h \neq 0}} \frac{(K_h u_h, u_h)}{(u_h, u_h)} \leq \lambda(K_h) \leq \max_{\substack{u_h \in \mathbb{R}^{N_h} \\ u_h \neq 0}} \frac{(K_h u_h, u_h)}{(u_h, u_h)} \leq \lambda_{max}(K_h) \overset{!}{\leq} \overline{\gamma}$$

$\Updownarrow$

which is equivalent to the so-called spectral equivalence inequalities

(11)' $$\underline{\gamma}(u_h, u_h) \leq (K_h u_h, u_h) \leq \overline{\gamma}(u_h, u_h) \quad \forall u_h \in \mathbb{R}^{N_h},$$

where $(\cdot,\cdot)$ denotes the Euclidean inner product in $\mathbb{R}^{N_h}$.

• $(K_h \underline{u}_h, \underline{u}_h) = a(u_h, u_h)$ $\begin{cases} \geq \mu_1 \|u_h\|_1^2 = ① \geq ? \\ \leq \mu_2 \underbrace{\|u_h\|_1^2}_{} = ② \leq ? \end{cases}$

$\forall \underline{u}_h \in \mathbb{R}^{N_h}$, $\underline{u}_h \overset{FE}{\underset{iso}{\longleftrightarrow}} u_h \in V_{0h}$

independent of the BVP!

① $\|u_h\|_1^2 = \int_\Omega (|\nabla_x u_h|^2 + u_h^2)\, dx = \sum_{r \in R_h} \int_{\delta_r} (|\nabla_x u_h|^2 + u_h^2)\, dx$ $\quad \bar\Omega = \cup_r \bar\delta_r$

mapping $\delta_r \leftrightarrow \Delta$

(12) $$= \sum_{r \in R_h} \int_\Delta \underbrace{\left(|J_{\delta_r}^{-T} \nabla_\xi u_h(x_{\delta_r}(\xi))|^2 + u_h^2(x_{\delta_r}(\xi)) \right)}_{\geq 0} |J_{\delta_r}|\, d\xi$$

(1) (mms) $\nabla_x = J_{\delta_r}^{-T} \nabla_\xi$

$$\geq \underline{c}_1 h^d \sum_{r \in R_h} \int_\Delta \Big(\overbrace{\sum_{\alpha \in A_r} u^{(r,\alpha)} p^{(\alpha)}(\xi)}^{u_h(x_{\delta_r}(\xi))} \Big)^2 d\xi$$

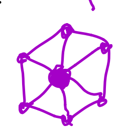
$$= \underline{c}_1 h^d \sum_{r \in R_h} \sum_{\alpha,\beta \in A_r \subseteq A = \{1,2,3\}\ \text{e.g.}} u^{(r,\alpha)} u^{(r,\beta)} \int_\Delta p^{(\alpha)}(\xi) p^{(\beta)}(\xi)\, d\xi$$

$$= \underline{c}_1 h^d \sum_{r \in R_h} \left(G_0^{(r)} \underline{u}^{(r)}, \underline{u}^{(r)} \right)_{\mathbb{R}^{|A_r|}}$$

$\underline{u}^{(r)} = [u^{(r,\alpha)}]_{\alpha \in A_r}$, $G_0^{(r)} = \left[\int_\Delta p^{(\alpha)} p^{(\beta)} d\xi \right]_{\alpha,\beta \in A_r}$, $G_0 = \left[\int_\Delta p^{(\alpha)} p^{(\beta)} d\xi \right]_{\alpha,\beta \in A}$

mass matrix of the master elem.

$$\geq \underline{c}_1 \lambda_{min}(G_0) h^d \sum_{r \in R_h} \sum_{\alpha \in A_r} (u^{(r,\alpha)})^2 \geq$$



$$\geq \underline{c}_1 \lambda_{min}(G_0) h^d (\underline{u}_h, \underline{u}_h), \text{ i.e.}$$

(13) $\underline{\gamma} = \underline{c}_E h^d$, with $\underline{c}_E = \mu_1 \underline{c}_1 \lambda_{min}(G_0)$

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② $$\|u_h\|_1^2 \overset{(12)}{=} \sum_{r\in R_h}{}' \int_\Delta \left(|J_{\delta_r}^{-T}\nabla_\xi u_h(x_{\delta_r}(\xi))|^2 + u_h^2(x_{\delta_r}(\xi))\right)|J_{\delta_r}|\,d\xi$$

$$\overset{(8)}{\le} \bar c_1 h^d \sum_{r\in R_h}{}' \int_\Delta \left(\nabla_\xi^T u_h \cdot J^{-1}_{\delta_r} J^{-T}_{\delta_r}\nabla_\xi u_h + u_h^2\right) d\xi$$

$$\le \bar c_1 h^d \sum_{r\in R_h}{}' \int_\Delta \left(\underbrace{\lambda_{max}(J^{-1}_{\delta_r} J^{-T}_{\delta_r})}_{\overset{(10)}{\le}\, c_3^2 h^{-2} \,\overset{!}{\ge}\, 1} |\nabla_\xi u_h|^2 + u_h^2(x_{\delta_r}(\xi))\right) d\xi$$

WLG, i.e. $h \le c_3$ (*)

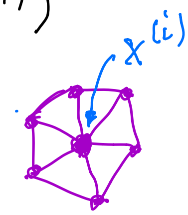
$$\le \bar c_1 h^d c_3^2 h^{-2} \sum_{r\in R_h}{}' \sum_{\alpha,\beta\in A_r}{}' u^{(r,\alpha)} u^{(r,\beta)} \underbrace{\int_\Delta \left(\nabla_\xi^T p^{(\alpha)} \nabla_\xi p^{(\beta)} + p^{(\alpha)} p^{(\beta)}\right) d\xi}_{= G_1^{(\alpha,\beta)}}$$

$$= \bar c_1 c_3^2 h^{d-2} \sum_{r\in R_h} \left(G_1^{(r)} \underline{u}^{(r)}, \underline{u}^{(r)}\right)$$

master element stiffness matrix for $-\Delta_\xi + I$: $G_1^{(r)} = [G_1^{(\alpha,\beta)}]_{\alpha,\beta\in A_r}$, $G_1 = [G_1^{(\alpha,\beta)}]_{\alpha,\beta\in A}$

$$\le \bar c_1 c_3^2 \lambda_{max}(G_1)\, h^{d-2} \sum_{r\in R_h}{}' \sum_{\alpha\in A_r} (u^{(r,\alpha)})^2$$

regular triangulation: $c_4 = \sup_{h\in\Theta} \max_{i\in\omega_h} |B_i|$



$$\le \bar c_1 c_3^2 c_4 \lambda_{max}(G_1)\, h^{d-2} (\underline{u}_h, \underline{u}_h) .$$

(14) $$\bar\gamma = \underbrace{\mu_2 \bar c_1 c_3^2 c_4 \lambda_{max}(G_1)}_{= \bar c_E} h^{d-2}$$

q.e.d.

E 3.5 Ex 3.5.

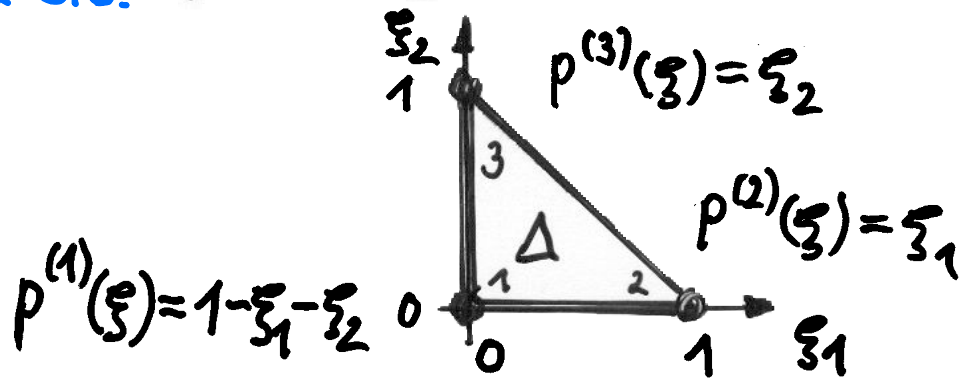
Ex. 3.3: $\underline{c}_1 = \frac{1}{2}\alpha_0^2 \sin\theta_0$, $\bar{c}_1 = \sqrt{3}/2$

Show that a regular family of triangular meshes in the sense of definition (4) is also a regular triangulation in the sense of Definition 3.3! Provide the constants $\underline{c}_1, \bar{c}_1, c_2$ and c_3!

$c_2 = ?$
$c_3 = ?$

E 3.6 Ex 3.6.

Compute $\lambda_{min}(G_0) = ?$ and $\lambda_{max}(G_1) = ?$ for Courant's element



and $\underline{\gamma} = ?$ and $\bar{\gamma} = ?$ for our model problem (2) on a regular triangular mesh!
<u>Hint:</u> Use the results of E 2.5!

E 3.7 Ex 3.7

Show that the eigenvalue estimates (11) are sharp with respect to the h-order, i.e.
$\exists\ \underline{c}_E', \bar{c}_E' = const > 0 :\ \underline{c}_E', \bar{c}_E' \neq c(h)$ and
$\lambda_{min}(K_h) \le \underline{c}_E' h^d$ and $\lambda_{max}(K_h) \ge \bar{c}_E' h^{d-2}$!
Therefore: $\lambda_{min} = O(h^d), \lambda_{max} = O(h^{d-2}), \varkappa(K_h) = O(h^{-2})$.

E 3.8 Ex 3.8

Show the spectral equivalence inequalities

$$\underline{c}_M h^d (\underline{u}_h, \underline{u}_h) \le (M_h \underline{u}_h, \underline{u}_h) = \|u_h\|^2_{L_2(\Omega)} \le \bar{c}_M h^d (\underline{u}_h, \underline{u}_h)$$

$$\forall\, \underline{u}_h = [u^{(i)}]_{i \in \omega_h} \longleftrightarrow u_h = \sum_{i \in \omega_h} u^{(i)} p^{(i)} \in V_{0h},$$

with the mass matrix $M_h = [\int_\Omega p^{(j)} p^{(i)} dx]_{i,j \in \omega_h}$.

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- **Remark:** The properties of the stiffness matrix K_h are very important for evaluating the accuracy and efficiency of **direct** and **iterative** **SOLVERS** for the system

$$K_h \underline{u}_h = \underline{f}_h$$

of finite element equations.

References:

[JL:2013] Chapter 5

[Steinbach:2008] = [5] (NuEPDE-Website)

CISM Courses on **Direct and Iterative Solvers** by U. Langer and M. Neumüller (2017)

Direct Solvers * = Part1DirectSolvers.pdf

Iterative Solvers * = Part2IterativeSolvers.pdf

Preconditioners * = Part3Preconditioners.pdf

Multigrid I * = Part4MultgridI.pdf

Multigrid II * = Part5MultgridII.pdf

www.numa.uni-linz.ac.at/Teaching/LVA/2019w/NumOpt/*