3.2.4. Generation of the FE Equations $K_h \underline{u}_h = \underline{f}_h$

a) Assembling of the Load Vector $\hat{\underline{f}}_h$

- Starting Point: $\forall k \in \omega_h$

$$f^{(k)} = \langle F, p^{(k)} \rangle - \sum_{i \in \gamma_h} \underset{\substack{\| \\ g_1(x^{(i)})}}{u_*^{(i)}} \, a(p^{(i)}, p^{(k)})$$

model problem from Subsection 3.2.1

$$= \underbrace{\int_\Omega f(x)\, p^{(k)}(x)\, dx}_{} + \underbrace{\underbrace{\int_{\Gamma_2} g_2 p^{(k)} ds}_{\text{2nd Kind BC}} + \underbrace{\int_{\Gamma_3} g_3 p^{(k)} ds}_{\text{3rd Kind BC}} - \underbrace{\sum_{i \in \gamma_h} g_1(x^{(i)})\, a(p^{(i)}, p^{(k)})}_{\text{inhom. 1st Kind BC}}}_{\text{will be considered later} \to c)}$$

- For the time being, we only consider the contribution of $\int_\Omega f p^{(k)} dx$ to $f^{(k)}$:

$$(6) \quad \hat{f}^{(k)} = \int_\Omega f(x)\, p^{(k)}(x)\, dx = \sum_{r \in B_k} \int_{\delta_r} f(x) \underset{\substack{\| \\ p^{(r,\alpha_k)}(x)}}{p^{(k)}(x)}\, dx$$

$k \in \overline{\omega}_h = \omega_h \cup \gamma_h$ $\qquad$ connectivity table $\boxed{r : k \leftrightarrow \alpha_k}$ $\qquad$ shape function

$$\hat{\underline{f}}_h = [\hat{f}^{(k)}]_{k \in \overline{\omega}_h}$$

In the FE practice, the components $\hat{f}^{(k)}$ will not be calculated nodel-wise via formula (6), but element-wise: $\longrightarrow$ **ASSEMBLING !**

- <u>Consider again our example CHIP:</u> $\hat{f}^{(8)}$

$$\hat{f}^{(8)} = \int_{\delta_8} f\, p^{(8)}\, dx + \int_{\delta_9} f\, p^{(8)} dx + \int_{\delta_{18}} f\, p^{(8)} dx + \int_{\delta_{19}} f\, p^{(8)} dx + \int_{\delta_{20}} f\, p^{(8)} dx$$

$\uparrow$

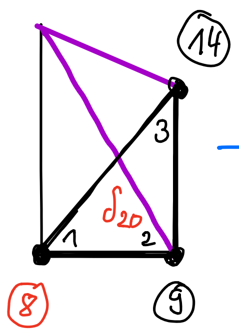
$B_8 = \{8, 9, 18, 19, 20\}$

<u>Loop over all elements:</u> $r = 1, \dots, 8, 9, \dots, 18, 19, \boxed{20}, 21, \dots, 24.$

$p^{(8)}(x)\big|_{\delta_{20}} =$

$= p^{(20,1)}(x)$

$= p^{(1)}(\xi_{\delta_{20}}(x))$



$\longrightarrow$ element Load vector

$$\underline{f}^{(20)} = \begin{bmatrix} f^{(20,1)} \\ f^{(20,2)} \\ f^{(20,3)} \end{bmatrix} \begin{array}{l} \longrightarrow \hat{f}^{(8)} \\ \longrightarrow \hat{f}^{(9)} \\ \longrightarrow \hat{f}^{(14)} \end{array}$$

assembling

$\boxed{r : i \longleftrightarrow \alpha}$

connectivity table

$$\int_{\delta_{20}} f(x)\, p^{(8)}(x)\, dx = \int_{\delta_{20}} f(x)\, p^{(20,1)}(x)\, dx = \int_{\Delta} f(x_{\delta_{20}}(\xi))\, p^{(1)}(\xi)\, |J_{\delta_{20}}|\, d\xi \approx$$

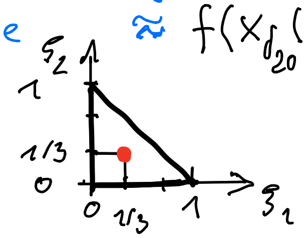
$\boxed{r = 20:\ \underset{\alpha}{1} \longleftrightarrow \underset{i}{8}}$

$\delta_{20} \longleftrightarrow \Delta$

quadrature rule

e.g.

$$\approx f(x_{\delta_{20}}(\xi_*))\, \underbrace{p^{(1)}(\xi_*)}_{=1/3}\, \underbrace{|J_{\delta_{20}}|}_{=2\ \mathrm{meas}\ \delta_{20}} \cdot \frac{1}{2} =: f^{(20,1)}$$



$\xi_* = \left(\frac{1}{3}, \frac{1}{3}\right)$ = centre of inertia

Ex. 3.4 Proof the relationship $(*)$ $\int_{\Delta} \varphi(\xi)\, d\xi = \varphi\left(\frac{1}{3}, \frac{1}{3}\right) |\Delta|\ \forall \varphi \in P_1$,

and show that $\xi_* = \left(\frac{1}{3}, \frac{1}{3}\right) \in \bar{\Delta}$ is the <u>only</u> point such that $(*)$ is valid!

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Algorithm: $\underline{\hat{f}}_h = ⑪$

$$\boxed{\begin{array}{l}
\underline{\text{for}}\ r = 1\ \underline{\text{step}}\ 1\ \underline{\text{until}}\ R_h\ \underline{\text{do}} \\
\quad \underline{\text{for}}\ \alpha = 1\ \underline{\text{step}}\ 1\ \underline{\text{until}}\ 3\ \underline{\text{do}} \\
\qquad \underline{\text{begin}} \\
\qquad * \text{ compute } f^{(r\alpha)} := f(x_{\delta_{r}}(\xi_\alpha))\, p^{(\alpha)}(\xi_\alpha)\, |J_{\delta_r}|\, \frac{1}{2} \\
\qquad * \text{ determine } i = i(r,\alpha) \qquad \boxed{r : \alpha \longleftrightarrow i = i(r,\alpha)} \\
\qquad * \text{ update } f^{(i)} := f^{(i)} + f^{(r\alpha)} \\
\qquad \underline{\text{end}} \\
\quad \underline{\text{endfor}} \\
\underline{\text{end for}}
\end{array}}$$

Theoretical representation of $\underline{\hat{f}}_h$ using the connectivity matrix C_r:

$$\underline{\hat{f}}_h = [\hat{f}^{(\kappa)}]_{\kappa \in \bar{\omega}_h} = \sum_{r \in R_h} C_r \underline{f}^{(r)},$$

with the connectivity matrices (= Boolean matrices):

$$C_r = \begin{array}{c} \\ \\ i_1 \\ \\ i_3 \\ \\ i_2 \\ \\ \\ \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & & \\ 1 & 0 & 0 \\ \vdots & & \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}_{\bar{N}_h \times 3} \end{array} = P_r \qquad \boxed{r : \alpha \text{ — } i}$$

$1 \longleftrightarrow i_1$
$2 \longleftrightarrow i_2$
$3 \longleftrightarrow i_3$

where $P_r = C_r : \mathbb{R}^{|A_r|=3} \longrightarrow \mathbb{R}^{\bar{N}_h}$ – prolongation operator,
$R_r = P_r^T : \mathbb{R}^{\bar{N}_h} \longrightarrow \mathbb{R}^{|A_r|=3}$ – restriction operator, $\bar{N}_h = |\bar{\omega}_h|$.

b) Assembling of the stiffness matrix $\hat{K}_h = [\hat{K}_{ij}]_{i,j\in\bar{\omega}_h}$

- **Starting point:** $\forall i,j \in \omega_h$

$$K_{ij} = a(p^{(j)}, p^{(i)}) = \begin{cases} 0, & \text{if } B_{ij} = B_i \cap B_j = \emptyset \\ \underbrace{\sum\limits_{r\in B_{ij}} \Big(\int\limits_{\delta_r} (\lambda \nabla^T p^{(j)} \nabla p^{(i)} + a p^{(j)} p^{(i)})\,dx}\; + \underbrace{\int\limits_{\Gamma_3 \cap \partial\delta_r} \varkappa p^{(j)} p^{(i)}\, ds_x}_{\text{3rd Kind BC}} & \end{cases}$$

model problem from Subsection 2.2.1

3rd Kind BC $\longrightarrow$ c)

- **FE technology for elementwise computing the contributions to K_{ij}:**

$$\hat{K}_{ij} = \sum_{r\in B_{ij}} \int\limits_{\delta_r} (\lambda \nabla^T p^{(j)} \nabla p^{(i)} + a p^{(j)} p^{(i)})\,dx \;\Rightarrow\; \hat{K}_h = [\hat{K}_{ij}]_{i,j\in\bar{\omega}_h}$$

$\forall i,j \in \bar{\omega}_h = \omega_h \cup \gamma_h$: $B_{ij} := B_i \cap B_j \neq \emptyset$

NOT componentwise, BUT elementwise !!

Example: $i=8,\ j=9 \Rightarrow B_{ij} = \{8,9,18,19,20\} \cap \{9,10,20\} = \{9,20\}$

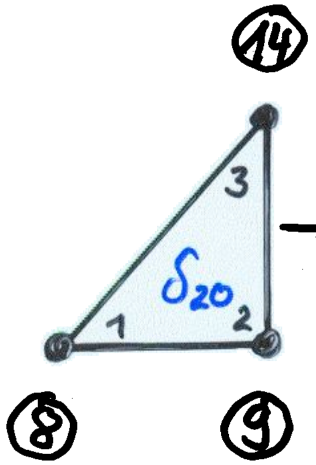
$\Gamma_3 \cap$, $\Gamma_3 \cap \partial\delta_{20} = \emptyset$

$$\hat{K}_{8,9} = \int\limits_{\delta_9} (\quad \ldots \quad)\,dx + \int\limits_{\delta_{20}} (\quad \ldots \quad)\,dx$$

Loop over all elements: $r = 1, \ldots, 8, 9, 10, \ldots, 19, \boxed{20}, 21, \ldots, 24;$

$\boxed{r: \alpha \leftrightarrow i}$

1	2	3
8	9	14



element stiffness matrix $K^{(20)} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}_{3\times 3}$ $\oplus$ $\begin{bmatrix} \hat{K}_{11} & & \\ & \hat{K}_{88} & \hat{K}_{89} \\ & & \end{bmatrix}_{\bar{N}_h \times \bar{N}_h}$

Assembling $\hat{K}_h$

$$\int_{\delta_{20}} \left(\lambda \frac{\partial p^{(9)}}{\partial x_1}\frac{\partial p^{(8)}}{\partial x_1} + \lambda \frac{\partial p^{(9)}}{\partial x_2}\frac{\partial p^{(8)}}{\partial x_2} + a\, p^{(9)} p^{(8)}\right) dx =$$

$r = 20$: $1 \leftrightarrow 8$, $2 \leftrightarrow 9$, $3 \leftrightarrow 14$ $\qquad \delta_{20} \longleftrightarrow \Delta$

$$= \int_{\delta_{20}} \left(\lambda \frac{\partial p^{(20,2)}}{\partial x_1}\frac{\partial p^{(20,1)}}{\partial x_1} + \lambda \frac{\partial p^{(20,2)}}{\partial x_2}\frac{\partial p^{(20,1)}}{\partial x_2} + a\, p^{(20,2)} p^{(20,1)}\right) dx =$$

$$= \int_{\Delta} \left(\lambda(x_{\delta_{20}}(\xi)) \frac{\partial p^{(2)}(\xi)}{\partial x_1}\frac{\partial p^{(1)}}{\partial x_1} + \lambda(\quad) \frac{\partial p^{(2)}(\xi)}{\partial x_2}\frac{\partial p^{(1)}(\xi)}{\partial x_2} + a(\cdot)\, p^{(2)} p^{(1)}\right) |J_{\delta_{20}}|\, d\xi$$

$$x = x_{\delta_{20}}(\xi) = J_{\delta_{20}}\,\xi + x^{(8)}$$

$$\nabla_x = J_{\delta_r}^{-T} \nabla_\xi, \quad \nabla_x = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \quad \nabla_\xi = \left(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}\right)^T$$

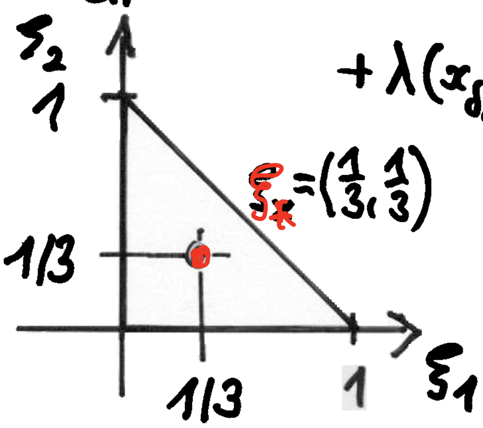
$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = J_{\delta_{20}}^{-1}\left(x - x^{(8)}\right), \quad J_{\delta_{20}}^{-1} = \frac{1}{|J_{\delta_{20}}|} \begin{bmatrix} x_2^{(14)} - x_2^{(8)} & -(x_1^{(14)} - x_1^{(8)}) \\ -(x_2^{(9)} - x_2^{(8)}) & x_1^{(9)} - x_1^{(8)} \end{bmatrix}$$

$$\nabla_x = J_{\delta_r}^{-T} \nabla_\xi \left\{ \begin{aligned} \frac{\partial}{\partial x_1} &= \frac{\partial}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1} + \frac{\partial}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_1} = \\ &= \frac{1}{|J_{\delta_{20}}|}\left[(x_2^{(14)} - x_2^{(8)})\frac{\partial}{\partial \xi_1} - (x_2^{(9)} - x_2^{(8)})\frac{\partial}{\partial \xi_2}\right] \\ \frac{\partial}{\partial x_2} &= \frac{1}{|J_{\delta_{20}}|}\left[-(x_1^{(14)} - x_1^{(8)})\frac{\partial}{\partial \xi_1} + (x_1^{(9)} - x_1^{(8)})\frac{\partial}{\partial \xi_2}\right] \end{aligned} \right.$$

$$p^{(2)}(\xi) = \xi_1, \quad \frac{\partial p^{(2)}}{\partial \xi_1} = 1, \quad \frac{\partial p^{(2)}}{\partial \xi_2} = 0,$$
$$p^{(1)}(\xi) = 1 - \xi_1 - \xi_2, \quad \frac{\partial p^{(1)}}{\partial \xi_1} = -1, \quad \frac{\partial p^{(1)}}{\partial \xi_2} = -1,$$

$$\overset{\text{QF}}{\approx} \Big\{ \lambda(x_{\delta_{20}}(\xi_*)) \left[\frac{1}{|J_{\delta_{20}}|}(x_2^{(14)} - x_2^{(8)}) \frac{1}{|J_{\delta_{20}}|}(x_2^{(9)} - x_2^{(8)})\right]$$

$$+ \lambda(x_{\delta_{20}}(\xi_*)) \left[\frac{1}{|J_{\delta_{20}}|}(x_1^{(8)} - x_1^{(14)}) \frac{1}{|J_{\delta_{20}}|}(x_1^{(14)} - x_1^{(9)})\right]$$



$$+ a(x_{\delta_{20}}(\xi_*))\, \frac{1}{3} \cdot \frac{1}{3} \Big\} |J_{\delta_{20}}| \cdot \frac{1}{2}$$

$$=: K_{1,2}^{(20)}$$

■ Algorithm: $\hat{K}_h := \mathbb{O}$

```
FOR r:=1 STEP 1 UNTIL R_h DO   (loop over all elements)
    FOR α:=1 STEP 1 UNTIL 3 DO
        FOR β:=1 STEP 1 UNTIL 3 DO
            BEGIN
                * compute    K^(r)_αβ := {...} |J_δr0| 1/2   for the model probl.
                * determine  i := i(r,α)                      r : α <-> i
                             j := j(r,β)                      r : β <-> j
                * update     K_ij := K_ij + K^(r)_αβ
            END
        ENDFOR
    ENDFOR
ENDFOR
```

■ Theoretical representation of $\hat{K}_h$ using the connectivity matrices C_r:

$$\hat{K}_h = [K_{ij}]_{i,j \in \overline{\omega}_h} = \begin{bmatrix} \equiv \end{bmatrix}_{\overline{N}_h \times \overline{N}_h} =$$

$$\overline{N}_h = N_h + \partial N_h$$

$$= \sum_{r \in R_h} C_r K^{(r)} C_r^T = \sum_{r \in R_h} P_r K^{(r)} R_r$$

with $P_r = C_r : \mathbb{R}^3 \to \mathbb{R}^{\overline{N}_h}$

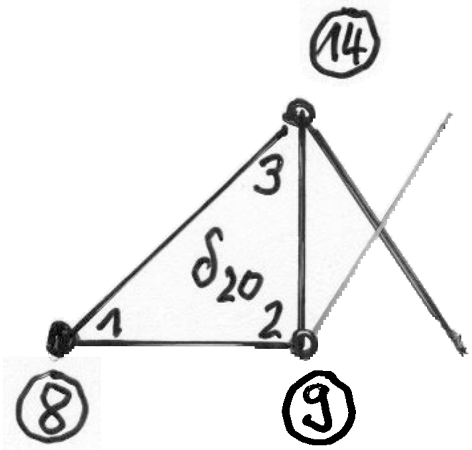
$R_r = C_r^T : \mathbb{R}^{\overline{N}_h} \to \mathbb{R}^3$

$$\boxed{r : \alpha \leftrightarrow i}$$

c) Incorporation of the Boundary Conditions:

- Natural BC: Γ_2 $(\to \langle F, \cdot \rangle)$, Γ_3 $(\to \langle F, \cdot \rangle, a(\cdot,\cdot))$
  - **inhomogeneous 2nd Kind BC:** $\int_{\Gamma_2} g_2 p^{(\alpha)} ds \xrightarrow{\oplus} f^{(\alpha)} \to \underline{\hat{f}}_h$

Again, the contribution to $\underline{f}_h$ will be generated elementwise (boundary-edge-wise), e.g. $9 \longrightarrow 14$



$$p^{(14)}(x) = p^{(20,3)}(x) = p^{(3)}(\xi_{\delta_{20}}(x))$$

$$p^{(9)}(x) = p^{(20,2)}(x) = p^{(2)}(\xi_{\delta_{20}}(x))$$

$x^{(9)} + s(x^{(14)} - x^{(9)})$

$$\int_{9}^{14} g_2 p^{(9)} ds_x = \int_0^1 g_2(\cancel{x^{(14)} + s(x^{(9)} - x^{(14)})}) p^{(9)}(\ldots) |x^{(9)} - x^{(14)}| ds \xrightarrow{\oplus} f^{(9)} \to \underline{\hat{f}}_h$$

$$\int_{9}^{14} g_2 p^{(14)} ds_x = \int_0^1 g_2(\cancel{x^{(14)} + s(x^{(9)} - x^{(14)})}) p^{(14)}(\ldots) |x^{(9)} - x^{(14)}| ds \xrightarrow{\oplus} f^{(14)} \to \underline{\hat{f}}_h$$

$$\overset{MP}{\underset{\text{Gauß 1}}{\approx}} g_2(\cancel{x^{(14)} + \tfrac{1}{2}(x^{(9)} - x^{(14)})}) \underbrace{p^{(14)}(x^{(14)} + \tfrac{1}{2}(x^{(9)} - x^{(14)}))}_{= 1/2} |x^{(9)} - x^{(14)}|$$

Define set $E_{2,h} := \{ e_2 \subset \partial\delta_r \cap \Gamma_2 : \text{inhom. 2nd Kind BC} \}$ of all element edges with inhomogeneous 2nd Kind BC:

```
FOR e_2 ∈ E_2h DO
    FOR α ∈ A_e2 ⊂ A = {1,2,3} DO
        * compute    f^(e2,α) := ∫_e2 g_2(x) p^(e2,α)(x) ds = (↑)
        * determine  i = i(r,α) = i(e_2,α)        r ← e_2
        * update     f^(i) := f^(i) + f^(e2,α)
    ENDFOR
ENDFOR
```

Inhomogeneous 3rd Kind BC:

$$\int_{\Gamma_3} g_3 p^{(\kappa)} ds \xrightarrow{\oplus} f^{(\kappa)} \longrightarrow \underline{\hat{f}}_h$$ analogous to 2nd Kind BC (1)

$$\int_{\Gamma_3} \alpha(x) p^{(i)}(x) p^{(\kappa)}(x) ds \xrightarrow{\oplus} \hat{K}_{\kappa i} \longrightarrow \hat{K}_h$$

Define $E_{3,h} := \{ e_3 \subset \partial\delta_r \cap \Gamma_3 : \text{3rd Kind BC} \}$

$$\boxed{\begin{array}{l}
\underline{\text{FOR}}\ e_3 \in E_{3,h}\ \underline{\text{DO}} \\
\quad \underline{\text{FOR}}\ \alpha \in A_{e_3} \subset A = \{1, 2, 3\}\ \underline{\text{DO}} \\
\quad\quad \underline{\text{FOR}}\ \beta \in A_{e_3} \subset A = \{1, 2, 3\}\ \underline{\text{DO}} \\
\quad\quad\quad * \text{ compute } K_{\alpha\beta}^{(e_3)} := \int_{e_3} \alpha(x) p^{(e_3,\alpha)}(x) p^{(e_3,\beta)}(x) ds \\
\quad\quad\quad\quad = \ldots \text{ (mms)} \\
\quad\quad\quad * \text{ determine} \\
\quad\quad\quad\quad i := i(e_3, \alpha) = i(r, \alpha) \\
\quad\quad\quad\quad j := j(e_3, \beta) = j(r, \beta) \\
\quad\quad\quad * \text{ update } \hat{K}_{ij} := \hat{K}_{ij} + K_{\alpha\beta}^{(e_3)} \\
\quad\quad \underline{\text{ENDFOR}} \\
\quad \underline{\text{ENDFOR}} \\
\underline{\text{ENDFOR}}
\end{array}}$$

Example CHIP: $E_{3,h} = \{ ①-②, ②-③, ③-④, ④-⑤ \}$

$\quad 1 \quad 2 \quad 1 \quad 2 \quad 1 \quad 2 \quad 1 \quad 2$

Essential BC = 1st Kind BC: $u = g_1$ on Γ_1 (V_{g_h}, V_{0h})

1st Version: Homogenization

- $u_j = g_1(x^{(j)})$ $\forall j \in \gamma_h := \overline{\omega}_h \setminus \omega_h$
- update the RHS: $f^{(i)} := f^{(i)} - \sum_{j \in \gamma_h} K_{ij}\, g_1(x^{(j)})$ $\forall i \in \omega_h$ $(\forall i \in \overline{\omega}_h)$
- cancel the columns with the indices $j \in \gamma_h$
- cancel the rows with the indices $i \in \gamma_h$

2nd Version: Penalty technique

$K_{ii} := 10^s$, $f^{(i)} := 10^s g_1(x^{(i)})$ $\forall i \in \gamma_h$

$\Rightarrow u_i = g_i - 10^{-s} \sum_{j \neq i}! K_{ij} u_j$ with s - suff. large!

this corresponds to 3rd Kind BC with $\alpha = 10^s$!

3rd Version: $K_{ji} = K_{ij} = \delta_{ij}$ $\forall i \in \gamma_h$ $\forall j \in \overline{\omega}_h$ and
$f^{(i)} = g_1(x^{(i)})$ $\forall i \in \gamma_h$

RESULT: System of FE equations for determining the unknown nodal values $u^{(i)}$, $i \in \omega_h$:

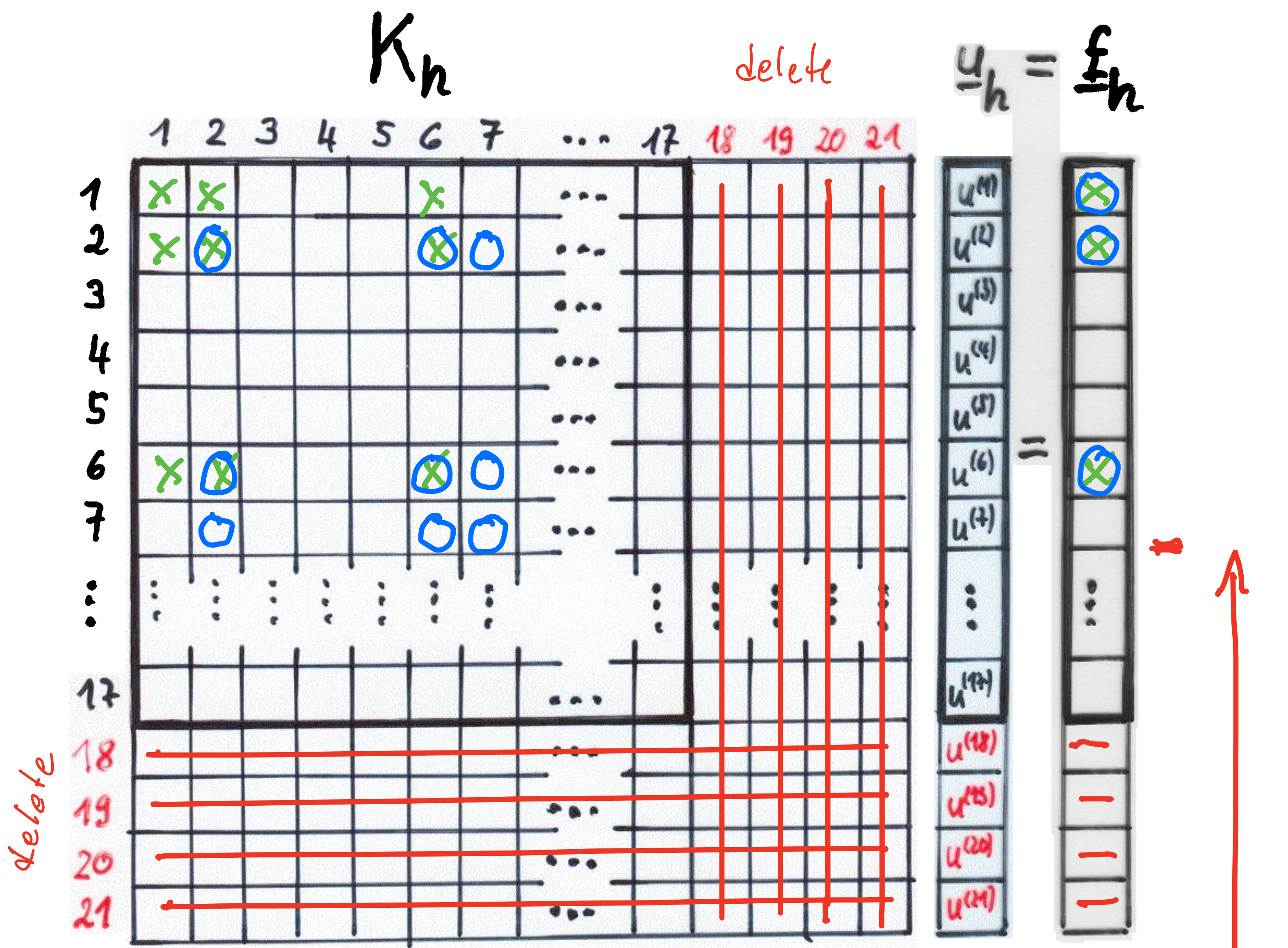
(i) Considering 1st Kind BC via 1st version:

$(\underline{4})_h$ Find $\underline{u}_h = [u^{(i)}]_{i \in \omega_h} \in \mathbb{R}^{N_h}$: $K_h \underline{u}_h = \underline{f}_h$ in $\mathbb{R}^{N_h}$

(ii) Considering 1st Kind BC via 2nd or 3rd version:

$(\underline{4})_h$ Find $\underline{u}_h = [u^{(i)}]_{i \in \overline{\omega}_h} \in \mathbb{R}^{\overline{N}_h}$: $K_h \underline{u}_h = \underline{f}_h$ in $\mathbb{R}^{\overline{N}_h}$

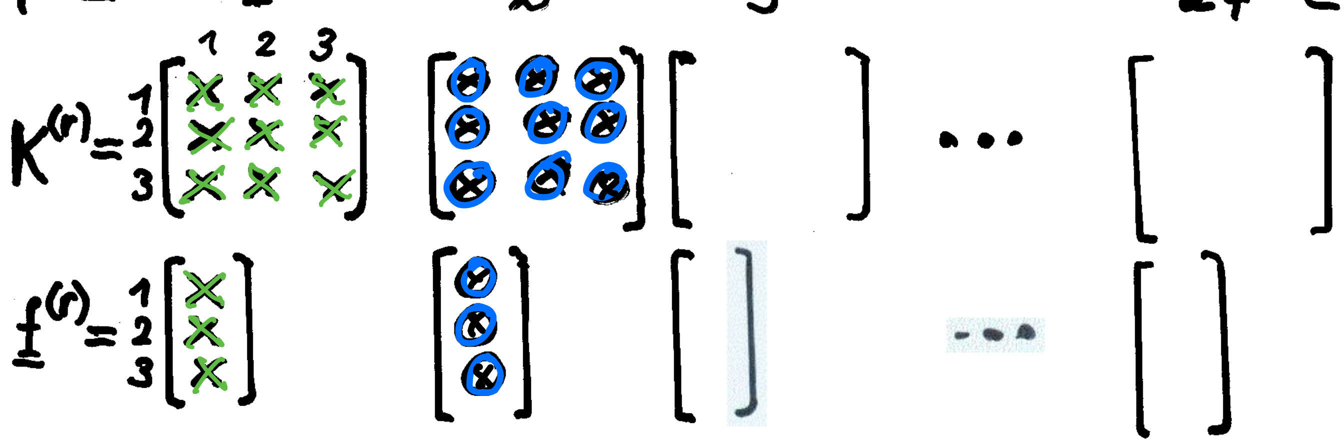
Example: CHIP



$\boxed{r : \alpha \leftrightarrow i}$ $\boxed{i : x_1^{(i)}, x_2^{(i)}}$

$$\sum_{1 \in \gamma_h = \{18, 19, 20, 21\}} K_{i1} g_1(x^{(1)})$$

$r = \quad 1 \quad 2 \quad 3 \quad \dots \quad 24 \in R_h$



<u>Taking into account the BC:</u>
2nd Kind : ~ homog. ($\to \underline{f}_h$)
3rd Kind : 1-2, 2-3, 3-4, 4-5 $\to \underline{f}_h$, $\to K_h$
1st Kind :