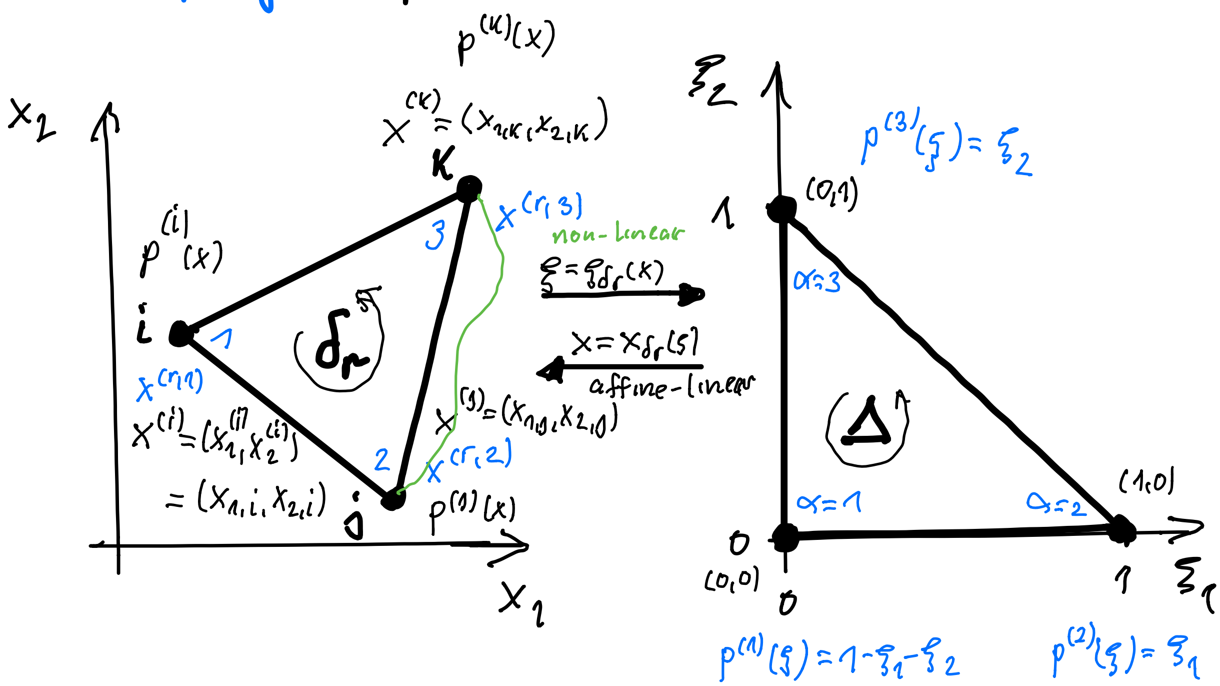
3.2.3. Definition of the FE Nodal Basis and of the $\bar{V}_h, \bar{V}_{0h}, \bar{V}_{gh}$ via Mapping Technique

■ **The basic principle** = mapping principle: $\delta_r \longleftrightarrow \Delta$

Local definition of the basis functions (trial, ansatz, test functions) via the shape functions ($=$ basis functions $|_{\delta_r}$), which are defined by mapping the shape functions of the master (reference or basis) element Δ to the element δ_r of the mesh.

■ **Mapping:** $\delta_r \longleftrightarrow \Delta$



$$\underset{S_{\delta_r}}{S(\delta_r)} = \operatorname{span}\{p^{(r,1)}, p^{(r,2)}, p^{(r,3)}\} = P_1(\delta_r) \leftrightarrows \underset{S_\Delta}{S(\Delta)} = \operatorname{span}\{p^{(1)}, p^{(2)}, p^{(3)}\} = P_1(\Delta)$$

arbitrary triangle δ_r, $r \in R_h$, of the mesh | **master triangle**

Affine Linear mapping: $x^{(i)} = (x_1^{(i)}, x_2^{(i)}) = (x_{1,i}, x_{2,i})$

$$x = x_{\delta_r}(\xi) = J_{\delta_r}\xi + x^{(i)}:$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_{1,j} - x_{1,i} & x_{1,k} - x_{1,i} \\ x_{2,j} - x_{2,i} & x_{2,k} - x_{2,i} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} x_{1,i} \\ x_{2,i} \end{pmatrix},$$

where $J_{\delta_r} = \frac{\partial x}{\partial \xi}$ = Jacobian of the mapping $x = x_{\delta_r}(\xi)$.

Then the following relation is valid:

$$(5)\quad \boxed{\begin{aligned} |J_{\delta_r}| &:= |\det J_{\delta_r}| = 2 \text{ meas } \delta_r \\ &\overset{\circlearrowleft}{=} (x_1^{(j)} - x_1^{(i)})(x_2^{(k)} - x_2^{(i)}) - (x_1^{(k)} - x_1^{(i)})(x_2^{(j)} - x_2^{(i)}) \\ &> 0 \end{aligned}}$$

Indeed, we have

$$\text{meas } \delta_r := \int_{\delta_r} dx = \int_{\Delta} \left|\frac{\partial x}{\partial \xi}\right| d\xi = \int_{\Delta} |J_{\delta_r}| d\xi = |J_{\delta_r}| \int_{\Delta} d\xi = |J_{\delta_r}| \cdot \frac{1}{2}.$$

$\boxed{\textbf{Ex. 3.3}}$ Prove the following inequalities:

$$\frac{1}{2}\alpha_0^2 \sin\theta_0 \, h^2 \le \frac{1}{2}\alpha_0^2 \sin\theta_r \, h_r^2 \le |J_{\delta_r}| \le \frac{\sqrt{3}}{2} h_r^2 \le \frac{\sqrt{3}}{2} h^2,$$

where h_r - largest edge of δ_r, and θ_r - smallest angle of δ_r.

$$\xi = \xi_{\delta_r}(x) = J_{\delta_r}^{-1}(x - x^{(i)}): \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{1}{|J_{\delta_r}|} \begin{pmatrix} x_{2,k} - x_{2,i} & -(x_{1,k} - x_{1,i}) \\ -(x_{2,j} - x_{2,i}) & x_{1,j} - x_{1,i} \end{pmatrix} \begin{pmatrix} x_1 - x_{1,i} \\ x_2 - x_{2,i} \end{pmatrix}$$

$$\downarrow$$

$$J_{\delta_r}^{-1} = \frac{\partial \xi}{\partial x}$$

Definition of the Basis Function:

→ trial and test functions:

$$p^{(i)}(x) = \begin{cases} p^{(r,\alpha)}(x), & x \in \bar{\delta}_r,\ r \in B_i \quad \text{(shape functions)} \\ 0, & \text{otherwise, i.e. } x \in \bar{\Omega} \setminus \underbrace{\bigcup_{r \in B_i} \bar{\delta}_r}_{= supp(p^{(i)})} \end{cases}$$

$\boxed{r : \alpha \leftrightarrow i}$ connectivity table!

with $B_i = \{ r \in R_h : x^{(i)} = (x_{1,i}, x_{2,i}) \in \bar{\delta}_r \}$

$\overset{e.g.}{\underset{i=8}{=}} \{8, 9, 18, 19, 20\}$,
CHIP

$$p^{(r\alpha)}(x) = p^{(\alpha)}(\xi_{\delta_r}(x)),\ x \in \bar{\delta}_r,\ p^{(\alpha)} \text{ - shape function on the master element}$$

Then: $\boxed{p^{(i)}(x^{(j)}) = \delta_{ij} \quad \forall i,j \in \bar{\omega}_h}$ → nodal basis!

Example: CHIP : $i=8$

→ see L12-04 = T12

Definition of V_h, V_{0h}, V_{gh}:

$$V_h := \{ v_h = \sum_{i \in \bar{\omega}_h} v^{(i)} p^{(i)}(x) \} \subset V = H^1(\Omega) \ ; \ v_h(x^{(i)}) = v^{(i)} \ \forall i \in \bar{\omega}_h,$$

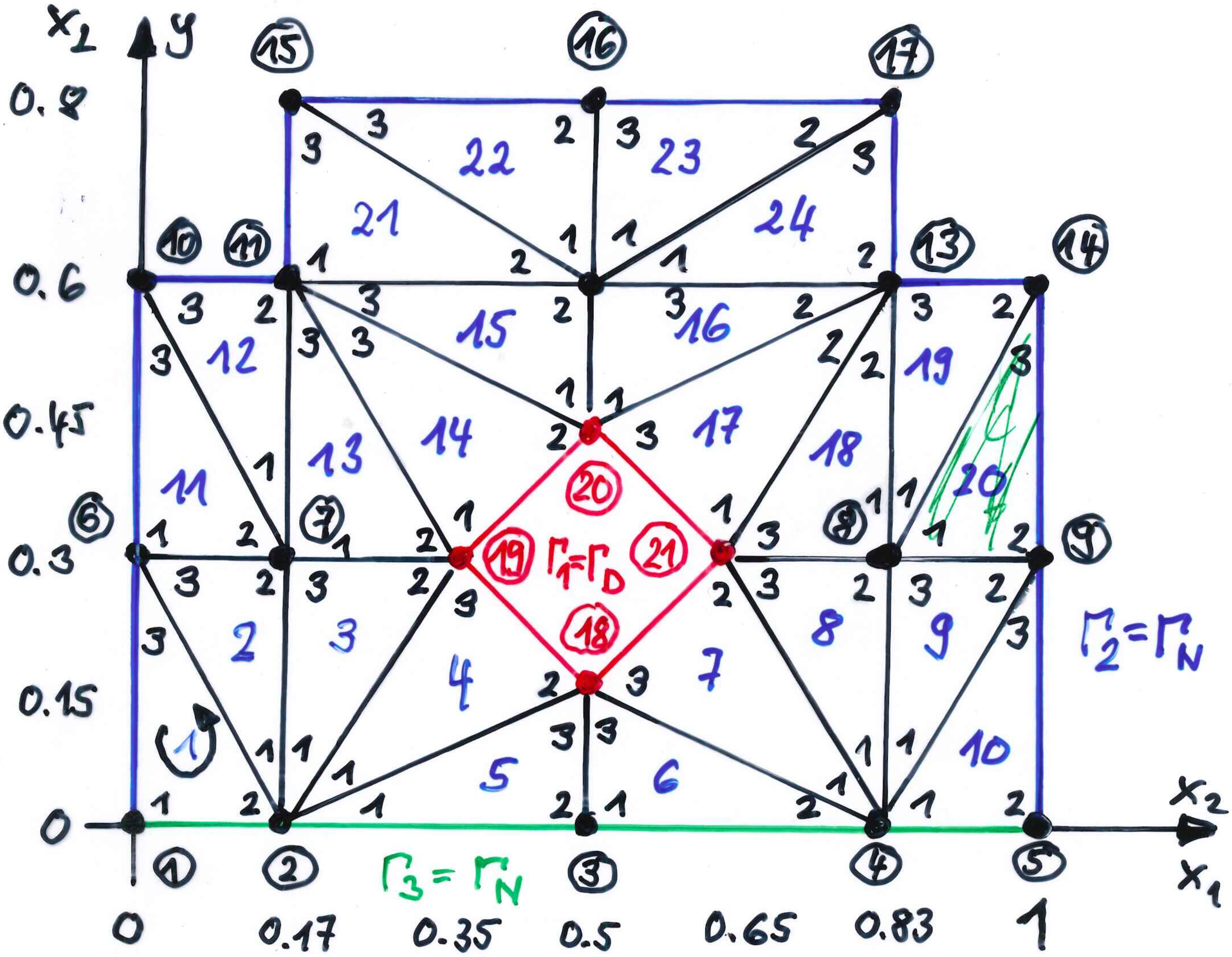
$$V_{0h} := \{ v_h = \sum_{i \in \omega_h} v^{(i)} p^{(i)}(x) \} \subset V_0, \text{ with } \omega_h := \{ i \in \bar{\omega}_h : x^{(i)} \notin \Gamma_1 \} = \bar{\omega}_h \setminus \gamma_h :$$

$$\Rightarrow v_h(x) = 0 \ \forall x \in \Gamma_1 \ \forall v_h \in V_{0h},$$

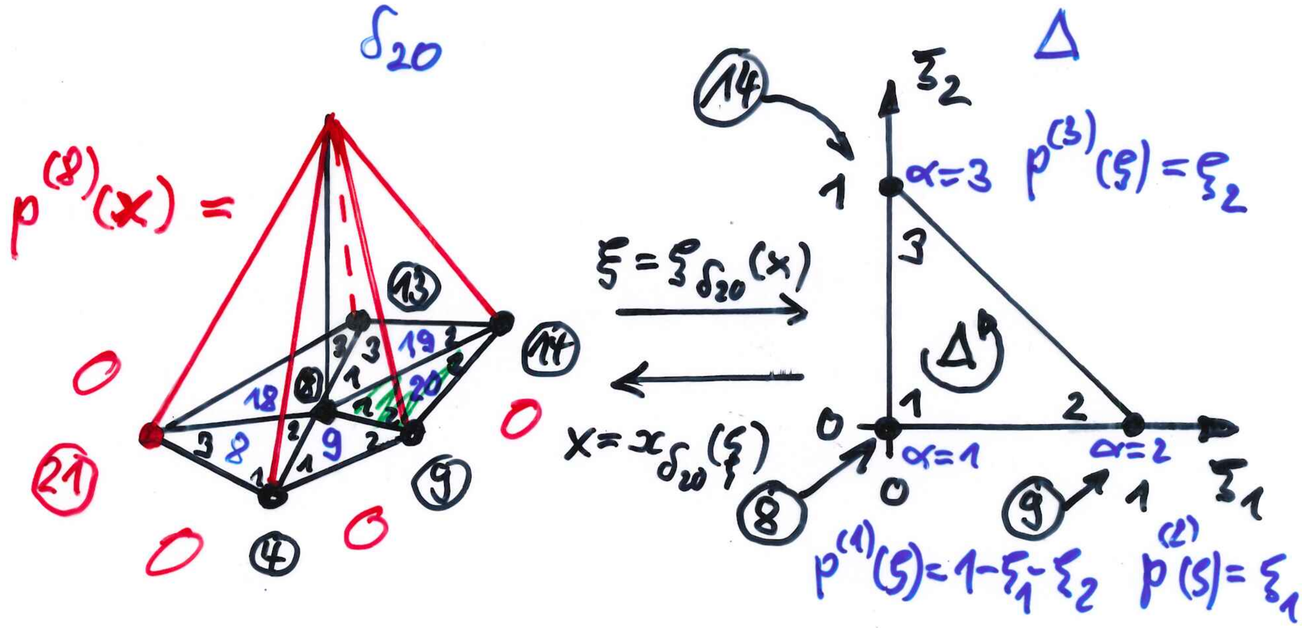
$$V_{gh} := \{ v_h = \sum_{i \in \omega_h} v^{(i)} p^{(i)}(x) + \sum_{i \in \gamma_h} g_1(x^{(i)}) p^{(i)}(x) \} \subset V_g :$$

$$\Rightarrow v_h(x) \overset{?}{=} g_1(x), \ \forall x \in \Gamma_1 \ \forall v_h \in V_{gh}, \text{ i.e. } g_1 \in span\{p^{(i)} : i \in \gamma_1\}.$$

$\approx$ crime?



■ <u>Example CHIP:</u> $i = 8$



$$p^{(8)}(x)\big|_{\delta_{20}} = p^{(20,1)}(x) := p^{(1)}(\xi_{\delta_{20}}(x))$$