3. THE GALERKIN FEM FOR ELLIPTIC BVP

Motivation: 1. Implicite time discretization (horizontal method of lines = Rothe's method) leads to a sequence of elliptic BVP $\Rightarrow$ Numerical solution of elliptic BVP is the basic task in the NuPDEs !!

2. Space-time methods = cG or dG FEM for IBVP: Find $u \in V_g(Q)$: $a(u,v) = \langle F, v \rangle \; \forall v \in V_0(Q)$, $Q = \Omega \times (0,T)$

3.1. FEM = GALERKIN-RITZ - Method with Special Trial (Ansatz) Functions

- **Recall** Lectures NuPDEs

Starting point: 1. Variational Formulation $(1.1) = (1)_{VF}$ or Minimization Problem $(1.4) = (1)_{MP}$

2. GALERKIN ~ RITZ - Method
$\rightarrow$ VF $\quad \rightarrow$ MP

$$V_h = \{ v_h = \sum_{\substack{i \in \overline{\omega}_h \\ i = 1,\dots,\overline{N}_h}} v^{(i)} p^{(i)}(x) \} \subset V = W_2^1(\Omega) = H^1(\Omega)$$

$$= \operatorname{span}\{ p^{(i)} : i \in \overline{\omega}_h \}$$

basic space for scalar 2nd-order PDEs

1 $\quad V_{gh} = g_h + V_{0h} = \{ v_h = \underbrace{\sum_{i \in \gamma_h} u_*^{(i)} p^{(i)}(x)}_{\substack{i = N_h+1,\dots,\overline{N}_h \\ -\shortparallel- |_{\Gamma_1} = g_1}} + \underbrace{\sum_{i \in \omega_h} v^{(i)} p^{(i)}(x)}_{\substack{i = 1,\dots,N_h \\ p^{(i)}|_{\Gamma_1} = 0 \; \forall i \in \omega_h}} \} \subset V_g \subset V$

$V_{gh} \subset V_g \cap V_h$, $\quad V_g = g + V_0$

V_g: hyperplane, lineare manifold

1 $\quad V_{0h} = V_h \cap V_0 = \{ v_h = \sum_{\substack{i \in \omega_h \\ i = 1,\dots,N_h}} v^{(i)} p^{(i)}(x) \} \subset V_0 \subset V$

4 $\quad = \operatorname{span}\{ p^{(i)} : i \in \omega_h \}$

V_0: infinite dimensional subspace

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(primal)
Variational Formulation — **Minimization Problem**

$(1)_{VF}$ $\boxed{u \in \bar{V}_g : a(u,v) = \langle F, v\rangle \ \forall v \in \bar{V}_0}$ $\Longleftrightarrow$ $\boxed{u \in \bar{V}_g : J(u) = \inf_{v \in \bar{V}_g} J(v)}$ $(1)_{MP}$

$a(\cdot,\cdot)$ symmetric and non-neg.

$\bar{V}_{gh} \subset \bar{V}_g$, $\bar{V}_{0h} \subset \bar{V}_0$ — $V_{gh} \subset \bar{V}_g = g + \bar{V}_0$

GALERKIN — RITZ

$(1)_h$ $\boxed{u_h \in \bar{V}_{gh} : a(u_h, v_h) = \langle F, v_h\rangle \ \forall v_h \in \bar{V}_{0h}}$ $\Longleftrightarrow$ $\boxed{u_h \in \bar{V}_{gh} : J(u_h) = \min_{v_h \in \bar{V}_{gh}} J(v_h)}$

$v_h = p^{(k)} \ \forall k \in \omega_h$

$$u_h = \sum_{i \in \omega_h}{}' u^{(i)} p^{(i)} + \underbrace{\sum_{i \in \gamma_h}{}' u_*^{(i)} p^{(i)}}_{= g_h \in \bar{V}_g \cap \bar{V}_h}$$

$\dfrac{\partial J(u_h)}{\partial u^{(k)}} = 0 \ \forall k \in \omega_h$

GALERKIN-RITZ-System = finite-dim. system of algebr. eqns

(2) Find $\underline{u}_h = [u^{(i)}]_{i \in \omega_h} \in \mathbb{R}^{N_h}$: $\displaystyle\sum_{i \in \omega_h}{}' u^{(i)} a(p^{(i)}, p^{(k)}) = \langle F, p^{(k)}\rangle - \sum_{i \in \gamma_h}{}' u_*^{(i)} a(p^{(i)}, p^{(k)})$ $\forall k \in \omega_h$

(3) $(\underline{1})_h$ $\qquad K_h \underline{u}_h = \underline{f}_h$

ansatz functions with global support

- **Main Difficulties of Classical GALERKIN-Methods:**
  1. Construction of $\bar{V}_{0h} \subset \bar{V}_0$ and $\bar{V}_{gh} \subset \bar{V}_g$,
  2. Generation of the GALERKIN-RITZ-System $(\underline{1})_h$
  3. Solution of $(\underline{1})_h$: Complexity = storage + #ops !
  4. Completeness of the family $\{\bar{V}_{0h}\}_{h \in \Theta}$ in $\bar{V}_0$

$$\lim_{h \in \Theta,\, |h| \to 0} \ \inf_{v_h \in \bar{V}_{0h}} \|u - v_h\|_{\bar{V}_0} = 0 \ \forall u \in \bar{V}_0$$

Example: Solve the VP ($V_g = \tilde{V}_0 = \tilde{V} = L_2(0,1)$):

$$(*) \text{ Find } u \in L_2(0,1): \int_0^1 u \cdot v \, dx = \int_0^1 f \cdot v \, dx \quad \forall v \in \tilde{V}_0$$

$$u \in V_g \quad : \quad a(u,v) = \langle F, v \rangle \quad \forall v \in \tilde{V}_0$$

$$u = f \in L_2(0,1)$$

$$V_h = \tilde{V}_{0h} = \tilde{V}_{gh} = \operatorname{span}\{x^i\}_{i=\overline{0,n-1}} \subset \tilde{V} = L_2(0,1)$$

$$= \left\{ \sum_{i=0}^{n-1} v^{(i)} x^i \right\}$$

$$K_{ij} = \int_0^1 x^{i+j} dx = \frac{1}{i+j+1} x^{i+j+1} \Big|_0^1, \text{ i.e}$$

$$K = \left[\frac{1}{i+j+1} \right]_{i,j=\overline{0,n-1}} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

= Hilbert-Matrix $= H_n$

Take $f(x) = \sin(\pi x)$ and solve the variational problem (*) numerically!

Have in mind that $\operatorname{cond}_2(H_n) := \frac{\lambda_{max}(H_n)}{\lambda_{min}(H_n)} = O\left(\frac{(1+\sqrt{2})^{4n}}{\sqrt{n}}\right)$,

i.e. $\operatorname{cond}_2(H_4) = 15 \cdot 10^3$, $\operatorname{cond}_2(H_5) = 4.8 \cdot 10^5$, $\operatorname{cond}_2(H_7) = 1.5 \cdot 10^7$

Idea: The basic idea to overcome the principle difficulties of the classical GALERKIN method (i.e. the use of basis (ansatz / tria) functions with global supports, e.g. polynomials) is due to Richard COURANT (1943):

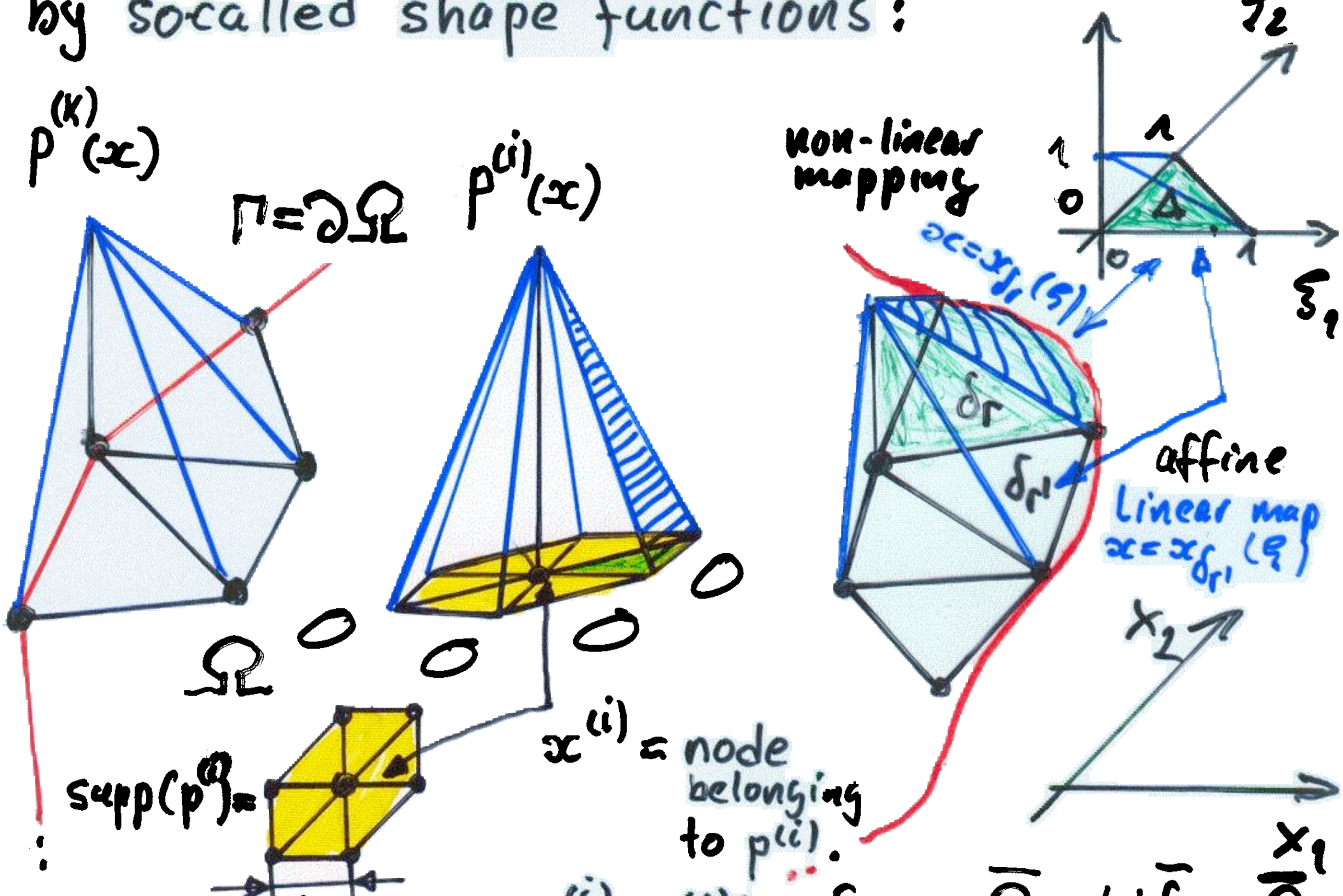
"Variational Methods for the Solution of Problems of Equilibrium and Vibrations"
Bull. Amer. Math. Soc., 49 (1943), pp. 1-23.

"If the variational problems contain derivatives not higher than the first order the method of finite difference can be subordinated to the Rayleigh-Ritz method by considering in the competition only functions ϕ which are linear in the meshes of a sub-division of our net into triangles formed by diagonals of the squares of the net. For such polyhedral functions the integrals become sums expressed by the finite number of values of ϕ in the net-points and the minimum conditions become our difference equations. Such an interpretation suggests a *wide generalization* which provides great flexibility and seems to have considerable practical value. Instead of starting with a quadratic or rectangular net we may consider from the outset any polyhedral surfaces with edges over an *arbitrarily chosen* (preferably triangular) *net*. Our integrals again become finite sums, and the minimum condition will be equations for the values of ϕ in the net-points. While these equations seem less simple than the original difference equations, we gain the enormous advantage of better adaptability to the data of the problem and thus we can often obtain good results with very little numerical calculation."

and was reinvented by engineers (J.H. ARGYRIS 1954 ff; M.J. TURNER, R.J. CLOUGH, H.C. MARTIN, L.J. TOPP, 1956, and others) in the mid 50ies:

⇓

use basis (ansatz / test (= trial)) functions $p^{(i)} = p^{(i)}(x)$ with local supports, where the $p^{(i)}$'s can be defined elementwise by socalled shape functions:



$p^{(i)}(x^{(j)}) = \delta_{ij}$

$\overline{\Omega}_h = \bigcup_r \overline{\delta}_r \approx \overline{\Omega}$

$\operatorname{diam} \operatorname{supp}(p^{(i)}) = O(h) \to 0$ nodal basis

$\operatorname{meas} \operatorname{supp}(p^{(i)}) = O(h^2) \to 0$

$\boxed{C^0\text{-elements}}$

$$\Rightarrow u_h(x) = \sum_{i \in \overline{\omega}_h} \underset{u_h(x^{(i)})}{\underset{\shortparallel}{u^{(i)}}} p^{(i)}(x) \underset{\text{mms}}{\underset{\uparrow}{\in}} W_2^1(\Omega) \cap C^0(\Omega)$$

Consequences of Courant's Idea:

1. Great flexibility in fulfilling the essential BC: (1)

2. Due to the locality of the support of the basis functions, the system matrix (= stiffness matrix) K_h is sparse. Obviously, we have

$$a(p^{(i)}, p^{(k)}) = 0 \text{ if } \operatorname{supp}(p^{(i)}) \cap \operatorname{supp}(p^{(k)}) = \emptyset.$$

$\Rightarrow$ (2), (3) !

3. The integrals for computing the entries of the stiffness matrix K_h and the load vector $\underline{f}_h$ can be computed elementwise: $\Rightarrow$ (2)

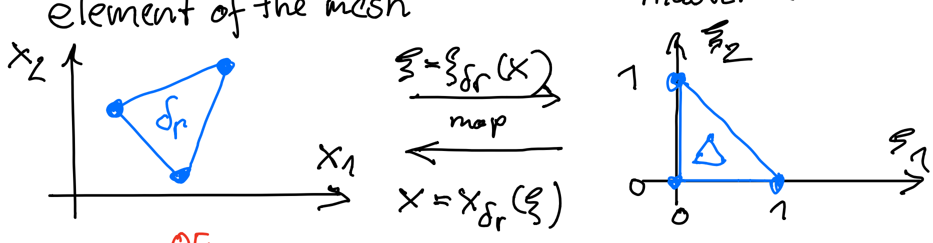
$$\overline{\Omega} \cong \overline{\Omega}_h = \bigcup_r \overline{\delta}_r \Rightarrow$$

quadrature formula

$$\int_{\Omega_h} [\dots]\, dx = \sum_r{}' \int_{\delta_r} [\dots]\, dx = \sum_r{}' \int_{\Delta} \underbrace{[\dots]\, |J_r|}_{=:\varphi_r(\xi)}\, d\xi \approx \text{ on } \Delta$$

element of the mesh

master element



$$\overset{QF}{\approx} \sum_r{}' \sum_\beta{}' w^{(\beta)} \varphi_r(\xi^{(\beta)})$$

4. Completeness of the family of FE spaces V_{0h} in V_0 ($\Delta \to$ [refined triangle] $\to \dots$) is relatively easy to verify! $\Rightarrow$ (4)

Notations:

Δ – master (reference) element,

δ_r – finite element, $r \in \mathcal{R}_h$,

$\Omega \subset \mathbb{R}^d$ – computational domain, $d = 1, 2, 3$,

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ – multiindex, $\alpha_i \in \{0, 1, \dots\} = \mathbb{N}_0$, $i = \overline{1,d}$,

$|\alpha| = \alpha_1 + \dots + \alpha_d$, $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \dots \cdot x_d^{\alpha_d}$, $\xi^\alpha = \dots$

$$\underset{\substack{\uparrow \\ \text{Shape}}}{S_\Delta} = \underset{\substack{\uparrow \\ \text{Form}}}{F_\Delta} = F(\Delta) = \Big\{ \sum_{\alpha \in A} v^{(\alpha)} p^{(\alpha)}(\xi) : \xi \in \overline{\Delta} \Big\}$$

$\uparrow$ index set

$$= \operatorname{span}\{ p^{(\alpha)} : \alpha \in A \}$$

$=$ space spanned by the shape functions,

$$P_k = \Big\{ \sum_{|\alpha| \le k} c_\alpha \xi^\alpha \Big\} = \text{space of polynomials of the degree } k,$$

$\cap$

$$Q_k = \Big\{ \sum_{0 \le \alpha_i \le k} c_\alpha \xi^\alpha \Big\} = \text{space of polynomials of the degree } k \text{ in every coordinate } \xi_i,\ i = 1, 2, \dots, d,$$

e.g. $P_1 = \operatorname{span}\{1, \xi_1, \xi_2\}$ – (affine) linear functions,

$\quad \overset{\cap}{Q_1} = \operatorname{span}\{1, \xi_1, \xi_2, \xi_1 \xi_2\}$ – bilinear functions.

Examples: Courant's triangle: $\Delta =$ 

$A = \{1, 2, 3\} = \{(0,0), (1,0), (0,1)\}$

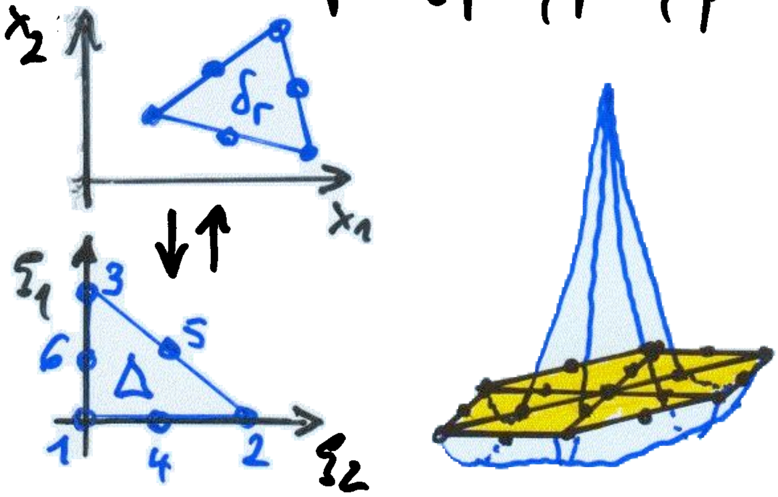
$S_\Delta = S(\Delta) = \operatorname{span}\{1 - \xi_1 - \xi_2, \xi_1, \xi_2\} = P_1 \subset Q_1$

Remark 2.1: Generalization to (Courant !)

1. Higher order basis (ansatz/test) functions on triangular elements:

(a) quadratic elements:

$$\mathcal{F}(\Delta) = \mathcal{P}_2 = \{a_0 + a_1\xi_1 + a_2\xi_2 + a_3\xi_1^2 + a_4\xi_1\xi_2 + a_5\xi_2^2\}$$
$$= \operatorname{span}\{p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}, p^{(5)}, p^{(6)}\}$$



nodal basis

$p^{(1)}(\xi) =$?

$p^{(2)}(\xi) =$?

....

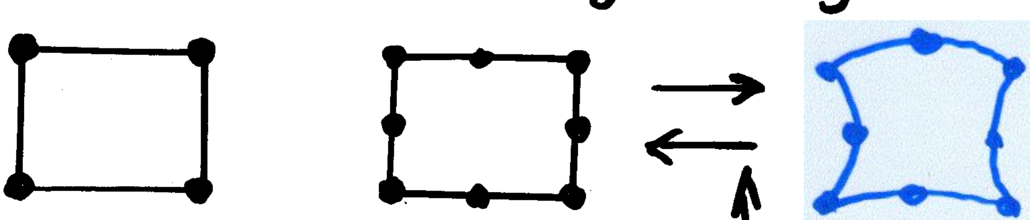
$$p^{(\alpha)}(\xi^{(\beta)}) = \delta_{\alpha,\beta}$$

(b) cubic elements: $\mathcal{F}(\Delta) = \mathcal{P}_3 = \operatorname{span}\{p^{(i)}\}_{i=\overline{1,10}}$

(c) general: Lagrangian elements of degree p

$$\mathcal{F}(\Delta) = \mathcal{P}_p \implies \boxed{C^0\text{-elements}}$$

2. Other 2D C^0-elements: e.g. rectangular elements



bilinear element $\mathcal{F}(\Delta) = Q_1$

SERENDIPITY element of 2nd order $\mathcal{F}(\Delta) \subset Q_2$

generalizations $\mathcal{P}_p \subset \mathcal{F}(\Delta) \subset Q_p$

isoparametric 2nd order SERENDIPITY element

$$x = x_{\delta_r}(\xi) = \sum_{\alpha\in A} x^{(\alpha)} p^{(\alpha)}(\xi)$$

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3. 3D C^0 - elements:

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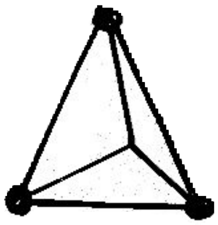
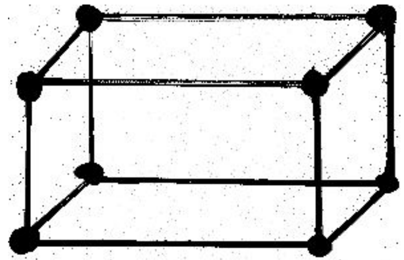


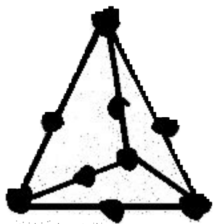
Linear tetrahedrial elements

$\mathcal{F}(\Delta) = \mathcal{P}_1$

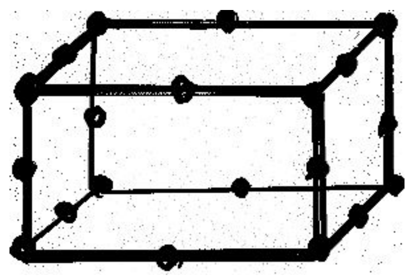


trilinear hexahedrial elements

$\mathcal{F}(\Delta) = Q_1$



quadratic tetrahedrial element



quadratic SERENDIPITY element

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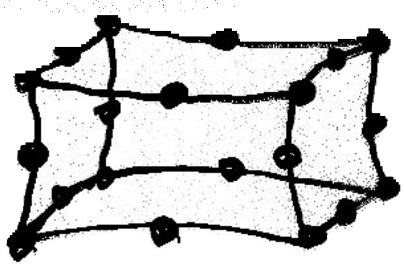
$\mathcal{P}_2 \subset \mathcal{F}(\Delta) \subset Q_2$

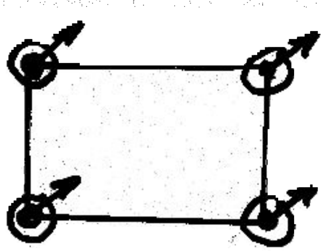
$$x = \sum_{\alpha \in A} x^{(a_\alpha)} p^{(\alpha)}(\xi) \longrightarrow \downarrow \uparrow$$



isoparametric quadratic SERENDIPITY element

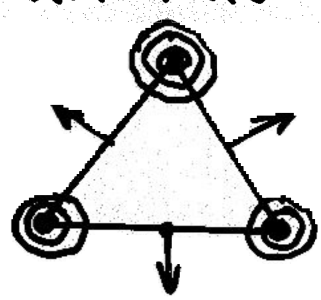
4. Higher Smoothness, e.g. C^1-elements for 4th-order PDEs

(a) HERMITE-Element : $\mathcal{F}(\Delta) = Q_3$



$$u_q^{(i)} \approx \begin{cases} u & \text{for } q=(0,0) \\ u_x & \text{for } q=(1,0) \\ u_y & \text{for } q=(0,1) \\ u_{xy} & \text{for } q=(1,1) \end{cases}$$

(b) ARGYRIS-ŽENYŽEK-Element : $\mathcal{F}(\Delta) = \mathcal{P}_5$



$u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$

$u_n = \frac{\partial u}{\partial n}$

21 dofs

L10 - Appendix 01 1943
Courant's famous paper

VARIATIONAL METHODS FOR THE SOLUTION OF PROBLEMS OF EQUILIBRIUM AND VIBRATIONS

R. COURANT

As Henri Poincaré once remarked, "solution of a mathematical problem" is a phrase of indefinite meaning. Pure mathematicians sometimes are satisfied with showing that the non-existence of a solution implies a logical contradiction, while engineers might consider a numerical result as the only reasonable goal. Such one sided views seem to reflect human limitations rather than objective values. In itself mathematics is an indivisible organism uniting theoretical contemplation and active application.

This address will deal with a topic in which such a synthesis of theoretical and applied mathematics has become particularly convincing. Since Gauss and W. Thompson, the equivalence between boundary value problems of partial differential equations on the one hand and problems of the calculus of variations on the other hand has been a central point in analysis. At first, the theoretical interest in existence proofs dominated and only much later were practical applications envisaged by two physicists, Lord Rayleigh and Walther Ritz; they independently conceived the idea of utilizing this equivalence for numerical calculation of the solutions, by substituting for the variational problems simpler approximating extremum problems in which but a finite number of parameters need be determined. Rayleigh, in his classical work—*Theory of sound*—and in other publications, was the first to use such a procedure. But only the spectacular success of Walther Ritz and its tragic circumstances caught the general interest. In two publications of 1908 and 1909 [39], Ritz, conscious of his imminent death from consumption, gave a masterly account of the theory, and at the same time applied his method to the calculation of the nodal lines of vibrating plates, a problem of classical physics that previously had not been satisfactorily treated.

Thus methods emerged which could not fail to attract engineers and physicists; after all, the minimum principles of mechanics are more suggestive than the differential equations. Great successes in applications were soon followed by further progress in the understanding of the theoretical background, and such progress in turn must result in advantages for the applications.

An address delivered before the meeting of the Society in Washington, D.C., on May 3, 1941, by invitation of the Program Committee; received by the editors June 16, 1942.

L10 - Appendix 02: FE-Example

negligible amount of numerical labor $S = .339$ and $c = -.11$. A refined attempt with the function

$$\phi = a(1 - x)[1 + \alpha(x - 3/4)y]$$

yielded $S = .340$ and $c = -.109$ with little more labor.

These results were checked with those obtained by our generalized method of finite differences where arbitrary triangular nets are permitted. The diagrams are self-explanatory. Unknown are the

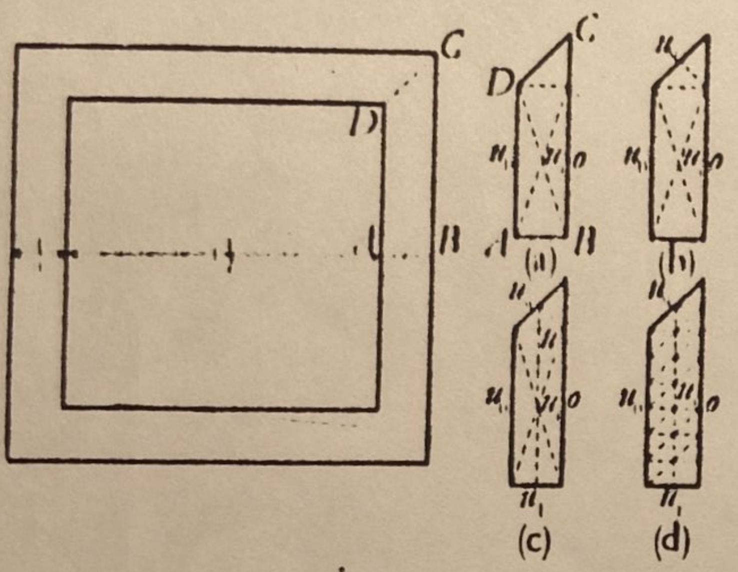


Fig. 2

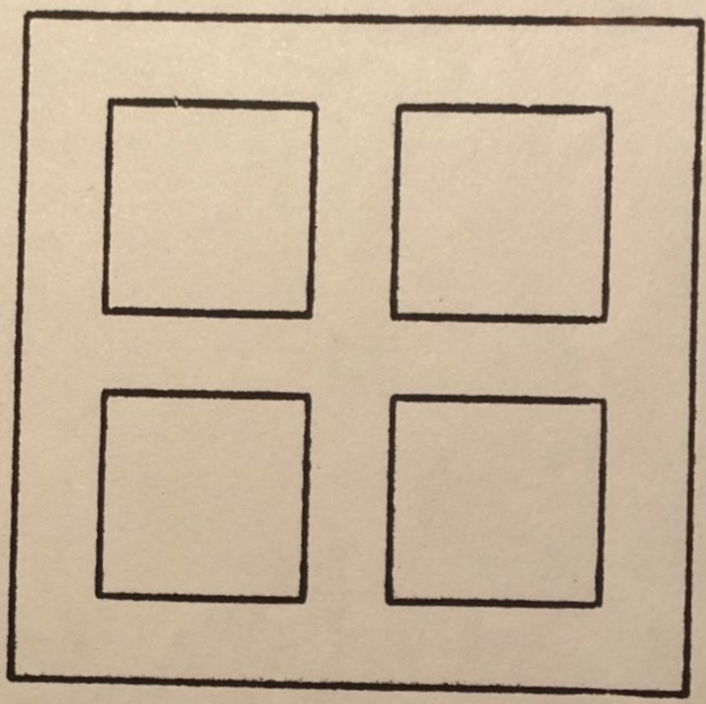


Fig. 3

net-point-values u_i, ($c = u_0$). In the net-triangles our functions were chosen as linear, so that the variational problem results in linear equations for the u_i. The results, easily obtainable, were: case (a) with two unknowns: $S = .344$, $u_0 = -.11$; case (b) with three unknowns: $S = .352$, $u_0 = -.11$; case (c) with five unknowns $S = .353$, $u_0 = -.11$; case (d) with nine unknowns, corresponding to the ordinary difference method $S = .353$, $u_0 = -.11$.

These results show in themselves and by comparison that the generalized method of triangular nets seems to have advantages. It was applied with similar success to the case of a square with four holes, and it is obviously adaptable to any type of domain, much more so than the Rayleigh-Ritz procedure in which the construction of admissible functions would usually offer decisive obstacles.

In a separate publication it will be shown how the method can be extended also to problems of plates and to other problems involving higher derivatives.

Of course, one must not expect good local results from a method