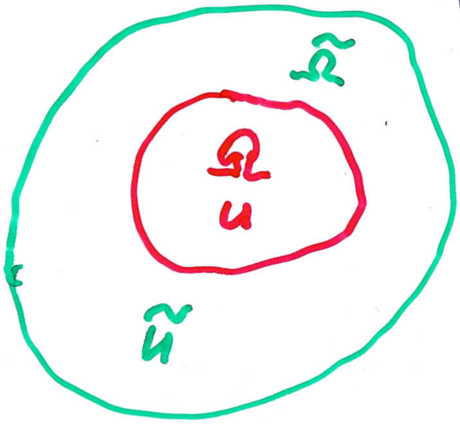
2.6. Extension Theorems

■ Extension Problem:

Let $B = B(\Omega)$ be a B-space of functions over Ω,
$\tilde{B} = B(\tilde{\Omega})$ be a B-space of functions over $\tilde{\Omega}$,
with $\overline{\Omega} \subset \tilde{\Omega}$ *



We look for some rule to construct an extension $\tilde{u} \in B(\tilde{\Omega})$ for some given function $u \in B(\Omega)$ such that

1. $u = \tilde{u}$ on Ω, i.e $u = \tilde{u}|_{\Omega}$,
2. $\|\tilde{u}\|_{B(\tilde{\Omega})} \leq c_E \|u\|_{B(\Omega)}$
   $\forall u \in B(\Omega)$, $u \longrightarrow \tilde{u} = Eu$

$\tilde{u}$ is called extension of u under keeping the class B.

■ History:

[1934] H. Whitney: $B = C^k$
[1941] R.M. Hestenes: $B = C^k$
$\longrightarrow$ see also Exercise 2.23 !
[1953] V.M. Babič: Generalization of the Hestenes extension procedure to $B = W_p^k$
[1961] A.P. Calderon: $B = W_p^k$
via integral representations

⋮

References!

[1981] S.G. Michlin: Konstanten in einigen Ungleichungen der Analysis. Teubner-Texte zur Mathematik, Leipzig 1981.

Theorem 2.22:

<u>Ass.:</u> $\Omega \subset \mathbb{R}^d$: $\ast \wedge$ Lip, $\Gamma = \partial\Omega$

$\tilde{\Omega} \subset \mathbb{R}^d$: $\ast \wedge \overline{\Omega} \subset \tilde{\Omega}$

$1 \le p \le \infty$, $k = 0, 1, \dots$

<u>St.:</u> Then there exists (not unique!) a Linear, bounded extension operator

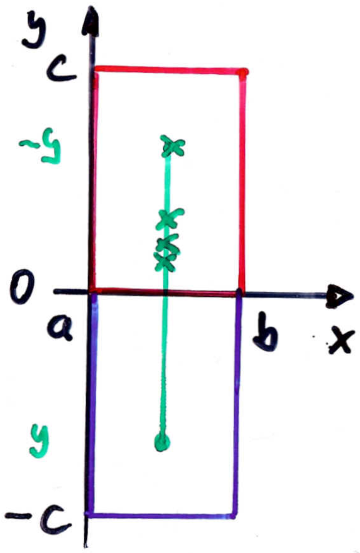
$$\Pi \in L(W_p^k(\Omega), \overset{\circ}{W}{}_p^k(\tilde{\Omega})),$$

i.e. $v = \Pi u$: 1. $v|_\Omega = u$

2. $\|v\|_{W_p^k(\tilde{\Omega})} = \|\Pi u\|_{W_p^k(\tilde{\Omega})} \le c_E \|u\|_{W_p^k(\Omega)}$.

<u>Proof:</u> see Lit., e.g. [Michlin] and Exercise 2.24 ∎

Exercise 2.23: Let



$\Omega = \{(x,y) \in \mathbb{R}^2 : a < x < b, 0 < y < c\}$

$\overline{\tilde{\Omega}} = \overline{\Omega} \cup \overline{\Omega}_-$, with

$\Omega_- = \{(x,y) \in \mathbb{R}^2 : a < x < b, 0 > y > -c\}$

$\Omega \subset \tilde{\Omega}$

$\exists!$ (mms) $k+1$ reals $\lambda_1, \dots, \lambda_{k+1}$:

$(-1)^\ell \lambda_1 + (-\frac{1}{2})^\ell \lambda_2 + \dots + (-\frac{1}{k+1})^\ell \lambda_{k+1} = 1$

$\ell = 0, 1, \dots, k$

Show that the Hestenes extension

$$v(x,y) = \begin{cases} u(x,y) & , 0 \le y \le c \\ \lambda_1 u(x,-y) + \dots + \lambda_{k+1} u(x, -\frac{y}{k+1}) & , -c \le y \le 0 \end{cases}$$

is really an extension of u under keeping the classes $B = C^k, W_2^\ell$, $\ell = 0, 1, \dots, k+1$.

2.7. Sobolev's Embedding Theorems

Def. 2.23:

X, Y – B-spaces: $X \subset Y$ (as sets)
The embedding operator $E : X \to Y$ assigns every element $u \in X$ the same element $u \in Y$ (after identification!).
The embedding is called continuous resp. compact iff the embedding operator E is continuous (*) resp. compact.

Lit.: Notation: $X \subset Y$ – continuous embedding,
$\hookrightarrow$ $X \hookrightarrow Y$ – compact embedding.
$\hookrightarrow$

Theorem 2.24: (Embedding for $\overset{\circ}{W}{}^1_p(\Omega)$)

Ass.: $\Omega \subset \mathbb{R}^d$ * ($\partial\Omega \in C^{0,1}$ is not required!)
$1 < p, q < \infty$

St.: 1. $\overset{\circ}{W}{}^1_p(\Omega) \hookrightarrow C(\overline{\Omega})$ if $p > d$,
i.e., for every function (equivalence class of functions) $u \in \overset{\circ}{W}{}^1_p(\Omega)$, there exists an equivalent function $u \in C(\overline{\Omega})$, and the embedding operator $E \in L(\overset{\circ}{W}{}^1_p(\Omega), C(\overline{\Omega}))$ and compact.

2. Let now $p \le d$. Then
a) $\overset{\circ}{W}{}^1_p(\Omega) \subset L_q(\Omega)$ if $q \le q_* := \frac{pd}{d-p} \wedge q < \infty$,
b) $\overset{\circ}{W}{}^1_p(\Omega) \hookrightarrow L_q(\Omega)$ if $q < q_*$.

3. Let $p \le d$ and $\Omega_m = \Omega \cap \mathcal{H}_m$:
$\operatorname{meas}_{\mathbb{R}^m}(\Omega_m) > 0$, where $\mathcal{H}_m$ is a m-dim. hyperplan: $d-p \le m$, $m \le d$, $m \ge 1$.
a) $\overset{\circ}{W}{}^1_p(\Omega) \subset L_q(\Omega_m)$ if $q \le q_* := \frac{pm}{d-p} \wedge q < \infty$,
b) $\overset{\circ}{W}{}^1_p(\Omega) \hookrightarrow L_q(\Omega_m)$ if $q < q_*$.

Theorem 2.25: (Embedding for $W_p^1(\Omega)$)

Ass.: $\Omega \subset \mathbb{R}^d$ * $\wedge$ Lip, $1 < p, q < \infty$,

St.: Statements 1. - 3. of Th. 2.24 remain valid if $\overset{\circ}{W}{}_p^1(\Omega)$ is replaced by $W_p^1(\Omega)$.

Theorem 2.26: (Embedding for $W_p^k(\Omega)$)

Ass.: $\Omega \subset \mathbb{R}^d$ * $\wedge$ Lip, $1 < p, q < \infty$,
$0 \le j < k$, $k = 1, 2, 3, \dots$

St.: 1. $W_p^k(\Omega) \hookrightarrow C^j(\bar{\Omega})$ if $(k-j)p > d$.

2. Let $(k-j)p \le d$.
Then the embeddings
$W_p^k(\Omega) \subset W_q^j(\Omega)$ resp. $\overset{\circ}{W}{}_p^k(\Omega) \subset \overset{\circ}{W}{}_q^j(\Omega)$
are
- continuous if $q \le q_* := \dfrac{pd}{d-(k-j)p} \wedge q < \infty$,
- compact if $q < q_*$.

The results on the embedding of $\overset{\circ}{W}{}_p^k(\Omega) \subset \overset{\circ}{W}{}_q^j(\Omega)$ don't need the assumption that Ω is Lip.

Remarks:

1. St. 3 of Th. 2.24 and Th. 2.25 can easily be generalized to $W_p^k(\Omega)$, i.e.
$W_p^k(\Omega) \subset W_q^j(\Omega_m)$ if (mms)
$W_p^k(\Omega) \hookrightarrow W_q^j(\Omega_m)$ if (mms)
2. All results are sharp!
For instance: $d=2$: $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$, BUT
$H^1(\Omega) \not\hookrightarrow C(\bar{\Omega})$!

Proofs: Lecture Notes "Numerik I", Sect. 3.8, pp. 80-86.