2.4. Poincaré's Inequality and Bramble-Hilbert's Lemma

- **Lemma 2.17:** (Lemma of Bramble and Hilbert, 1971)

Ass.: $\Delta \subset \mathbb{R}^d \neq \emptyset \wedge$ Lip. $\langle \partial\Delta \in C^{0,1})$
$1 \le p < \infty$, $k \in \{0,1,\dots\}$, $\sigma \in (0,1]$ fixed
$\ell(\cdot) \in [W_p^{k+\sigma}(\Delta)]^* : \ell(q) = 0 \;\forall q \in P_k := \{\sum_{|\alpha| \le k} c_\alpha x^\alpha\}$.

St.: Then there exists a constant $c_B = \text{const} > 0$:

$$(13) \quad |\ell(u)| \le c_B |u|_{W_p^{k+\sigma}(\Delta)} \quad \forall u \in W_p^{k+\sigma}(\Delta),$$

where $c_B = c(\Delta) \|\ell\|_*$ with

$$\|\ell\|_* \ge \sup_{v \in W_p^{k+\sigma}(\Delta)\setminus\{0\}} \frac{|\ell(v)|}{\|v\|_{W_p^{k+\sigma}(\Delta)}} =: \|\ell\|_{[W_p^{k+\sigma}(\Delta)]^*},$$

$c(\Delta) = \text{const}(\Delta, p, k, \sigma) > 0$ only depends on the Poincaré constant c_p for Δ.

Proof:

- $\ell(u) = \ell(u) + \underset{\overset{\shortparallel}{0}}{\ell(q)} \overset{\text{linear}}{=} \ell(u+q) \quad \forall q \in P_k \;\forall u \in W_p^{k+\sigma}(\Delta)$
- $|\ell(u)| = |\ell(u+q)| \le \|\ell\|_* \|u+q\|_{W_p^{k+\sigma}(\Delta)} \quad \forall q \in P_k \;\forall u \in W_p^{k+\sigma}(\Delta)$

$$\Longrightarrow \quad |\ell(u)| \le \|\ell\|_* \underbrace{\inf_{q \in P_k} \|u+q\|_{W_p^{k+\sigma}(\Delta)}}_{=: \|u\|_{W_p^{k+\sigma}(\Delta)|P_k}} \quad \forall q \in P_k \;\forall u \in W_p^{k+\sigma}(\Delta)$$

$$=: \|u\|_{W_p^{k+\sigma}(\Delta)|P_k} \overset{!}{\lesssim} |u|_{W_p^{k+\sigma}(\Delta)}$$

- We now show that

$$(14) \quad \inf_{q \in P_k} \|u+q\|_{W_p^{k+\sigma}(\Delta)} \le c(\Delta) |u|_{W_p^{k+\sigma}(\Delta)} \quad \forall u \in W_p^{k+\sigma}(\Delta)$$

by means of the Poincaré inequality $(12)_{\omega = \Omega = \Delta}$.
Then we are done, i.e. (13) is proved!

- **Wlg** we prove (14) for $p=2$ (Hilbert space case that we need later), $K=1$ and $\sigma=1$, but the proof technique can easily be generalized to the general case given in Lemma 2.17.

OK, we now show that

$$(14)_{p=2,K=1,\sigma=1} \quad \inf_{q\in P_1} \|u+q\|_2 \le C(\Delta)|u|_2 \ \forall u\in H^2(\Delta)$$

- $\|u+q\|_2^2 = \|u+q\|_1^2 + |u|_2^2 \ \forall u\in H^2(\Delta)=W_2^2(\Delta)\ \forall q\in P_1$,

i.e. it is enough to consider $\partial_i = \partial^{(0,\dots,1,\dots,0)} = \frac{\partial}{\partial x_i}$

$$\|u+q\|_1^2 = \|u+q\|_0^2 + \sum_{i=1}^d \|\partial_i(u+q)\|_0^2$$

$$\overset{\uparrow}{\le} C_P\Big\{\Big|\int_\Delta (u+q)\,dx\Big|^2 + \sum_{i=1}^d \int_\Delta |\partial_i(u+q)|^2 dx\Big\} + \sum_{i=1}^d \|\partial_i(u+q)\|_0^2$$

$$(12)\quad \boxed{\int_\Delta |u|^2 dx \le C_P\Big\{\Big(\int_\Delta u\,dx\Big)^2 + \sum_{i=1}^d \int_\Delta |\partial_i u|^2 dx\Big\} \quad (\text{Poincaré})}$$

$$= C_P\Big(\int_\Delta (u+q)\,dx\Big)^2 + (C_P+1)\sum_{i=1}^d \|\partial_i(u+q)\|_0^2$$

$$\overset{(12)}{\le} C_P\Big(\int_\Delta (u+q)\,dx\Big)^2 + (C_P+1)C_P\sum_{i=1}^d \Big\{\Big(\int_\Delta \partial_i(u+q)\,dx\Big)^2 +$$

$$+ \sum_{j=1}^d \int_\Delta \big(\underbrace{\partial_j(\partial_i(u+q))}\big)^2 dx\Big\}$$

$$\partial_j\partial_i = \partial_{ji} = \frac{\partial^2}{\partial x_j \partial x_i} = \partial_{ij}$$

$$= C_P\Big(\underbrace{\int_\Delta (u+q)\,dx}_{=0}\Big)^2 + (C_P+1)C_P\sum_{i=1}^d \Big(\underbrace{\int_\Delta \partial_i(u+q)\,dx}_{=0}\Big)^2 + (C_P+1)C_P|u|_2^2$$

$\exists q\in P_1$?

$\forall u\in H^2(\Delta)\ \forall q\in P_1$

- For every fixed $u \in H^2(\Delta)$ $\exists!\, q(x) = a_0 + \sum_{i=1}^{d} a_i x_i \in P_1$:

$$(13) \quad \boxed{\begin{aligned} \int_\Delta q(x)\,dx &= -\int_\Delta u(x)\,dx \\ \int_\Delta \partial_i q(x)\,dx &= -\int_\Delta \partial_i u(x)\,dx, \quad i=\overline{1,d} \end{aligned}}$$

Indeed, (13) is nothing but a linear system of algebraic equations for determing the coefficients $a_0, a_1, \dots, a_d$ of $q(x) = a_0 + a_1 x_1 + \dots + a_d x_d$ for given $u \in H^2(\Delta)$:

$$(13) \quad \boxed{\begin{aligned} a_0 |\Delta| + \sum_{i=1}^{d} a_i \int_\Delta x_i\,dx &= -\int_\Delta u(x)\,dx \\ a_i |\Delta| &= -\int_\Delta \partial_i u(x)\,dx, \quad i=\overline{1,d} \end{aligned}}$$

(13) has obviously a unique solution. Therefore, for all (fixed) $u \in H^2(\Delta)$ $\exists!\, q \in P_1$:

$$\|u+q\|_1^2 \le (c_p+1)c_p |u|_2^2.$$

Thus, we have

$$\inf_{q \in P_1} \|u+q\|_2^2 \le ((c_p+1)c_p+1)|u|_2^2 \quad \forall u \in H^2(\Delta),$$

i.e. $c(\Delta) = \sqrt{(c_p+1)c_p+1}$. **q.e.d.**

- **Remark:** $W_2^{1+1} \longmapsto W_p^{k+\sigma}$

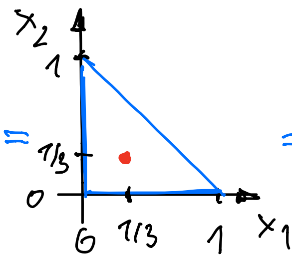
Here you need the Poincaré-like inequality (cmns by Th. 2.13):

$$\int_\Delta |u|^p\,dx \le c_p \left\{ \left|\int_\Delta u(x)\,dx\right|^p + |u|^p_{W_p^\sigma(\Delta)} \right\}.$$

Exercise 2.18: A first application of Bramble-Hilbert's lemma.

Let us consider the quadrature rule

$$\int_\Delta u(x)\,dx \approx u(x^*)\,|\Delta| \quad \text{with} \quad x^* = \left(\tfrac{1}{3}, \tfrac{1}{3}\right) \text{ and}$$

$\Delta =$ 

$= \{x = (x_1, x_2) \in \mathbb{R}^{d=2} : 0 < x_2 < 1 - x_1,\ 0 < x_1 < 1\}$.

Show that there exists a constant $c = \text{const} > 0$:

$$\left| \int_\Delta u(x)\,dx - u(x^*)\,|\Delta| \right| \le c\,|u|_2 \quad \forall u \in H^2(\Delta).$$

<u>Hint:</u> Use the continuous embedding of $H^2(\Delta)$ in $C(\bar{\Delta})$ for $d = 2$, see Theorem 2.26, i.e. $\exists\, c_E = \text{const} > 0$:

$$\|u\|_{C(\bar\Delta)} := \max_{x \in \bar\Delta} |u(x)| \le c_E \|u\|_{H^2(\Delta)} \quad \forall u \in H^2(\Delta),$$

and consider the linear functional

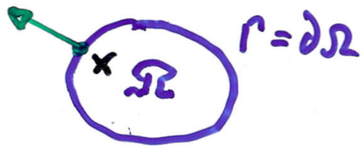
$$\ell(u) := \int_\Delta u(x)\,dx - u(x^*)\,|\Delta|.$$

2.5. The Formula of Integration by Parts and other Integration Formulas

- <u>The main formula of the Differential and Integral Calculus (DIC):</u>

  - $d = 1$

$$\int_a^b w'(x)\,dx = w(x)\Big|_a^b = w(b) - w(a) \quad \forall w \in C^1[a,b]$$



  - $d \in \mathbb{N}$

(14) $$\int_\Omega \partial_i w\,dx = \int_\Gamma w \cdot n_i\,ds \quad \forall w \in C^1(\overline{\Omega}),$$

where $n = n(x) = (n_1(x), \dots, n_d(x))^T$ – exterior unit normal,
$|n| = 1$, $n_i(x) = \cos \measuredangle(n(x), \vec{x}_i)$.

  - Inserting $w = u \cdot v$ with $u, v \in C^1(\overline{\Omega})$ into (14) and using the product rule give the classical formula of integration by parts:

(15) $$\int_\Omega \partial_i u \cdot v\,dx = -\int_\Omega u\,\partial_i v\,dx + \int_\Gamma u v n_i\,ds \quad \forall u, v \in C^1(\overline{\Omega})$$

  - <u>Lemma 2.18:</u>

<u>Ass.:</u> $\Omega \subset \mathbb{R}^d$ $\dagger \wedge$ Lip, $d \in \mathbb{N}_0$ (formal)
<u>St.:</u> Then the main formula of the DIC is also valid for functions $w \in W_1^1(\Omega)$:

(14) $$\boxed{\int_\Omega \partial_i w\,dx = \int_\Gamma w \cdot n_i\,ds \quad \forall w \in W_1^1(\Omega)}$$

<u>Proof</u> via "Closure Principle"!

$$W_1^1(\Omega) = \overline{C^1(\bar{\Omega})}^{\|\cdot\|_{W_1^1(\Omega)}} \ni w \xleftarrow[m\to\infty]{W_1^1(\Omega)} w_m \in C^1(\bar{\Omega}).$$

Trace Theorem (→ see $(9)_0$ on T2-07) gives

$$\|\underset{\gamma_0 w}{\overset{\|}{w}} - \underset{\gamma_0 w_m}{\overset{\|}{w_m}}\|_{L_1(\Gamma)} \le c\,\|w - w_m\|_{W_1^1(\Omega)} \xrightarrow[m\to\infty]{} 0$$

The classical formula yields

$$\int_\Omega \partial_i w_m \, dx = \int_\Gamma w_m \cdot n_i \, ds \qquad m\to\infty$$

$$\downarrow \qquad\qquad \downarrow$$

$$\int_\Omega \partial_i w \, dx = \int_\Gamma w \cdot n_i \, ds$$

$$\left| \int_\Gamma w \cdot n_i \, ds - \int_\Gamma w_m n_i \, ds \right| \le \int_\Gamma |w - w_m| \cdot |n_i| \, ds$$

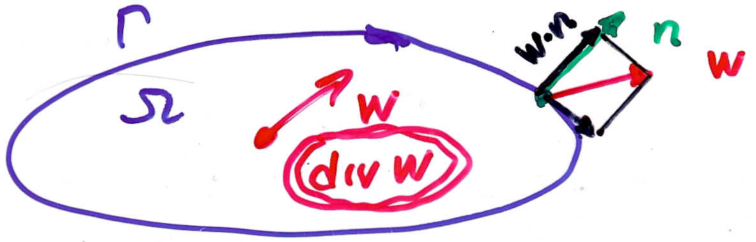
$$\le \underbrace{\|n_i\|_{L_\infty(\Gamma)}}_{\le 1} \|w - w_m\|_{L_1(\Gamma)} \le c\,\|w - w_m\|_{W_1^1(\Omega)} \xrightarrow{m\to\infty} 0$$

q.e.d.

■ <u>Gauss' Integration Theorem (balance identity):</u>

Let $w = (w_1, \ldots, w_d)^T$ a vector field with $w_i \in W_1^1(\Omega)$.
(14) immediately yields Gauss' Integration Theorem:

$$(16) \quad \int_\Omega \operatorname{div} w \, dx = \sum_{i=1}^d \int_\Omega \partial_i w_i \, dx = \sum_{i=1}^d \int_\Gamma w_i n_i \, ds = \int_\Gamma w \cdot n \, ds$$



Further Direct Consequences from (14):

1. Formula of Integration by Parts (IbyP):

(15) $$\int_\Omega \partial_i u \cdot v \, dx = -\int_\Omega u \partial_i v \, dx + \int_\Gamma u v n_i \, ds$$

$$\forall u \in W_p^1(\Omega), \ \forall v \in W_q^1(\Omega), \ \frac{1}{p} + \frac{1}{q} = 1, \ 1 < p < \infty$$

Proof: Set $w = u \cdot v \in W_1^1(\Omega)$ in (14). ■

2. The 1st Green's Formula for $(-\Delta)$:

(16') $$\int_\Omega \nabla^T u \cdot \nabla v \, dx = -\int_\Omega \Delta u \cdot v \, dx + \int_\Gamma \frac{\partial u}{\partial n} \cdot v \, ds$$

$$\forall u \in W_2^2(\Omega) \ \forall v \in W_2^1(\Omega)$$

Proof: Set $w = \partial_i u \cdot v \in W_1^1(\Omega)$ in (14) $\wedge \sum_{i=1}^{d}$ ■

3. The 2nd Green's Formula for $(-\Delta)$:

(16'') $$\int_\Omega (\Delta u \cdot v - u \Delta v) \, dx = \int_\Gamma \frac{\partial u}{\partial n} v \, ds - \int_\Gamma u \frac{\partial v}{\partial n} \, ds \quad \forall u, v \in H^2(\Omega).$$

Proof: follows immediately from (16') changing u and v ■

4. The 1st Green's Formula for Δ^2:

(16''') $$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega \Delta^2 u \cdot v \, dx - \int_\Gamma \partial_n \Delta u \cdot v \, ds + \int_\Gamma \Delta u \, \partial_n v \, ds$$

$$\forall u \in W_2^4(\Omega) \ \forall v \in W_2^2(\Omega), \ \partial_n = \partial/\partial n$$

Proof: $\int_\Omega \partial_i^2 u \, \partial_i^2 v \, dx$ two times IbyP ■

5. The IbyP-formula (15) yields (mms)

$$\int_\Omega \mathrm{curl}(u) \cdot v \, dx = \int_\Omega u \cdot \mathrm{curl}(v) \, dx - \int_\Gamma (u \times n) \cdot v \, ds.$$

$$\forall u, v \in H(\mathrm{curl}) \cap [C^1(\bar{\Omega})]^d \quad (d=3)$$

■ IbyP-Formula and Traces of H(div) Functions:

- <u>Starting Point:</u> $\forall q = (q_1, \ldots, q_d)^T \in [C^1(\bar{\Omega})]^d, v \in C^1(\bar{\Omega})$:

$$\int_\Omega \operatorname{div} q \cdot v \, dx = -\int_\Omega q \cdot \nabla v \, dx + \int_\Gamma q \cdot n \cdot v \, ds \text{ , i.e.}$$

$$(17) \quad \int_\Gamma q \cdot n \cdot v \, ds = \int_\Omega (\operatorname{div} q \cdot v + q \cdot \nabla v) \, dx \quad \forall q \in C^1(\bar{\Omega})^d, \forall v \in C^1(\bar{\Omega})$$

$$\overset{!!}{=} \langle \gamma_n q, \gamma_0 v \rangle_{H^{-1/2} \times H^{1/2}}, \quad \gamma_n q \in (H^{1/2}(\Gamma))^*, \quad \gamma_0 v \in H^{1/2}(\Gamma)$$

closure principle $\downarrow \; \downarrow$ $q \in H(\operatorname{div}), v \in H^1(\Omega)$

$$H^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))^*$$

- <u>Theorem 2.19</u> (H(div)-trace theorem)

There exists a unique continuous linear operator

$$\gamma_n \in L(H(\operatorname{div}), H^{-1/2}(\Gamma))$$

such that

$\gamma_n q(x) = q(x) \cdot n(x) \quad \forall x \in \Gamma \quad \forall q \in H(\operatorname{div}) \cap [C^1(\bar{\Omega})]^d$,

and, $\forall q \in H(\operatorname{div})$ and $\forall v \in H^1(\Omega)$, we have

$v=1$

$$(17) \quad \langle \gamma_n q, \gamma_0 v \rangle_{H^{-1/2} \times H^{1/2}} = \int_\Omega (\operatorname{div} q \cdot v + q \cdot \nabla v) \, dx$$

<u>Proof:</u> Following the closure principle, it remains to prove <u>continuity</u> (=?) on a smooth and dense subset: Let $q \in H(\operatorname{div}) \cap [C^1(\bar{\Omega})]^d$:

$$(18) \quad \|\gamma_n q\|_{H^{-1/2}(\Gamma)} = \sup_{w \in H^{1/2}(\Gamma)} \frac{\int_\Gamma q \cdot n \cdot w \, ds}{\|w\|_{H^{1/2}(\Gamma)}} \le$$

$$\overset{(17)}{\le} c_e \sup_{v \in H^1(\Omega)} \frac{\int_\Omega (\operatorname{div} q \cdot v + q \cdot \nabla v) \, dx}{\|v\|_{H^1(\Omega)}} \overset{\text{Cauchy}}{\le} c_e \sup_{v \in H^1(\Omega)} \frac{\|q\|_{H(\operatorname{div})} \|v\|_{H^1(\Omega)}}{\|v\|_{H^1(\Omega)}}$$

$$= c_e \|q\|_{H(\operatorname{div})}$$

(9)e $\|v\|_{H^1(\Omega)} \le c_e \|w\|_{H^{1/2}(\Gamma)}$, $\gamma_0 v = w$

Closure Principle q.e.d.

$H^{-1/2}(\Gamma) = \gamma_n H(\mathrm{div})$

- **Theorem 2.20:** ($H(\mathrm{div})$ inverse trace theorem)
Let $q_n \in H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))^*$. Then there exists a $q \in H(\mathrm{div})$ such that

(19) $\gamma_n q = q_n$ and $\|q\|_{H(\mathrm{div})} \le c_e \|q_n\|_{H^{-1/2}(\Gamma)}$.

If q_n satisfies $\langle q_n, 1\rangle = 0$, then there exits an extension $q \in H(\mathrm{div})$ such that $\mathrm{div}\, q = 0$.

Proof: Consider the weak solution of the Neumann problem

(20) $\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = q_n & \text{on } \Gamma. \end{cases}$

Since $q_n \in H^{-1/2}(\Gamma)$, we get from Lax-Milgram that $\exists !\, u \in H^1(\Omega)$: (mms)

(21) $\|\nabla u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} \le c\, \|q_n\|^2_{H^{-1/2}(\Gamma)}$.

Now, setting $q = \nabla u$, we observe

1) $\mathrm{div}\, q = \mathrm{div}\, \nabla u = \Delta u \overset{(20)}{=} u \in L_2(\Omega)$ (weak!)

2) $\|q\|_0^2 + \|\mathrm{div}\, q\|_0^2 = \|\nabla u\|_0^2 + \|u\|_0^2 \overset{(21)}{\le} c\, \|q_n\|^2_{H^{-1/2}(\Gamma)}$.

If q_n: $\langle q_n, 1\rangle = 0$, then consider

(22) $\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = q_n & \text{on } \Gamma \end{cases}$ $\exists (!)$ LaM $\wedge$ $\langle q_n, 1\rangle = 0$.

Again, set $q = \nabla u$ yielding $\mathrm{div}\, q = \mathrm{div}\, \nabla u = \Delta u \overset{(22)}{=} 0$.

q.e.d.

- **Exercise 2.21:** Let $\Omega_1, \dots, \Omega_m$ be a non-overlapping domain decomposition of $\overline{\Omega} = \cup \overline{\Omega}_i$, with $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$.

Let $q_i \in H(\mathrm{div}, \Omega_i)$: $\gamma_{n_i, \Gamma_{ij}} q_i = \gamma_{n_i, \Gamma_{ij}} q_j$ $\forall \Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$, $i,j = \overline{1,m}$.

Then $q := \{q = q_i \text{ on } \Omega_i,\ i = \overline{1,m}\} \in H(\mathrm{div})$.