2.3. Sobolev's Norm Equivalence Theorem and Some Useful Inequalities (cf. also Th. 1.3)

Theorem 2.13:

Ass.:
- $\Omega \subset \mathbb{R}^d$ * $\wedge$ Lip
- $1 \le p < \infty$
- $k = 0,1,2,\dots;\ \sigma \in (0,1],\quad s = k+\sigma$
- Let $f_i : W_p^{k+\sigma}(\Omega) \longrightarrow \mathbb{R}^1_0 := [0,\infty),\ i = 1,\dots,l$ be a system of semi-norms: $\forall i = \overline{1,l}$

  a) $\exists\, c_i = const > 0 : 0 \le f_i(u) \le c_i \|u\|_{W_p^s(\Omega)}\ \forall u \in W_p^s(\Omega)$

  b) $\left.\begin{array}{l} f_i(v) = 0\ \forall i = \overline{1,l} \\ v \in P_k := \{\sum\limits_{|\alpha|\le k} c_\alpha x^\alpha\} \end{array}\right\} \Rightarrow v \equiv 0.$

St.: Then $\exists\ \underline{c}, \overline{c} = const > 0$:

(10) $$\underline{c}\, \|u\|^*_{W_p^s(\Omega)} \le \|u\|_{W_p^s(\Omega)} \le \overline{c}\, \|u\|^*_{W_p^s(\Omega)} \quad \forall u \in W_p^s(\Omega)$$

$|||\cdot||| = \|\cdot\|^*$

$$\|u\|^*_{W_p^s(\Omega)} := \Big(\sum_{i=1}^{l} f_i^p(u) + |u|^p_{W_p^s(\Omega)}\Big)^{1/p} \simeq \|u\|^*_{W_p^s(\Omega)},$$

delete!

$$\|u\|_{W_p^s(\Omega)} := \Big(\sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha u|^p dx + |u|^p_{W_p^s(\Omega)}\Big)^{1/p},$$

$$|u|^p_{W_p^{k+\sigma}(\Omega)} := \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{d+p\sigma}}\, dx\, dy, \quad \sigma \in (0,1),$$

$$|u|^p_{W_p^{k+1}(\Omega)} := \sum_{|\alpha|=k+1} \int_\Omega |\partial^\alpha u(x)|^p dx, \quad \sigma = 1,$$

$|\cdot|_{W_p^{k+\sigma}(\Omega)}$ – standard semi-norm in $W_p^{k+\sigma}(\Omega)$:
$\mathrm{Ker}\, |\cdot|_{W_p^{k+\sigma}} = P_k$

black board

Proof for $\sigma = 1$ and $p = 2$ on the black board, but the proof is also valid for the general case!

$\|\cdot\|_{W_2^{k+1}} = \|\cdot\|_{k+1},\quad |\cdot|_{W_2^{k+1}(\Omega)} = |\cdot|_{k+1}$ etc.

a) $$\|u\|^*_{k+1} = \left(\sum_{i=1}^{l}(f_i(u))^2 + |u|^2_{k+1}\right)^{1/2}$$
$$\leq \left(\sum_{i=1}^{l} c_i^2\|u\|^2_{k+1} + |u|^2_{k+1}\right)^{1/2}$$
$$\leq \left(\sum_{i=1}^{l} c_i^2 + 1\right)^{1/2}\|u\|_{k+1}, \text{ i.e.}$$

$$\underline{c}\,\|u\|^*_{k+1} \leq \|u\|_{k+1} \quad \forall u \in H^{k+1}(\Omega) = W_2^{k+1}(\Omega),$$

with $\underline{c} = \left(\sum_{i=1}^{l} c_i^2 + 1\right)^{-1/2}$.

b) The proof of the right-hand side (rhs) of the norm equivalence inequalities (10) is unfourtunately NOT constructive and non-trivial ($\to$ by contradiction). For the proof, we need RELLICH's theorem of the compact embedding ($\hookrightarrow$) of $H^{k+1}(\Omega)$ into $H^k(\Omega)$ or more general (see also embedding theorems in Section 2.7):

$$\boxed{W_p^{k+\sigma}(\Omega) \hookrightarrow W_p^k(\Omega)}$$

- $\underline{\text{ASSUME}}$ that

$$\nexists\ \bar{c} = \text{const} > 0 : \ \|u\|_{k+1} \leq \bar{c}\,\|u\|^*_{k+1} \quad \forall u \in \underset{\underset{\displaystyle H^{k+1}(\Omega)}{\shortparallel}}{W_2^{k+1}(\Omega)}.$$

Then

$$\exists \{u_n\}_{n\in\mathbb{N}} \subset H^{k+1}(\Omega) : \quad n \leq \frac{\|u_n\|_{k+1}}{\|u_n\|^*_{k+1}}, \quad n = 1, 2, \ldots$$
$$u_n \neq 0$$

- Consider the sequence

$$v_n = \frac{u_n}{\|u_n\|_{k+1}} \Longrightarrow \quad 1)\ \|v_n\|_{k+1} = 1$$

$$2)\ \|v_n\|^*_{k+1} = \frac{\|u_n\|^*_{k+1}}{\|u_n\|_{k+1}} \leq \frac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

Therefore, relation 2), i.e. $\|v_n\|^*_{k+1} \to 0$, implies that

a) $|v_n|^2_{k+1} := \sum_{|\alpha|=k+1} \|\partial^\alpha v_n\|_0^2 \longrightarrow 0$ for $n \to \infty$,

b) $f_i(v_n) \longrightarrow 0$ for $n \to \infty$, $\forall i = 1,2,\dots,l$.

- On the other hand:

$\{v_n\}$ * in $W_2^{k+1}(\Omega) \Longrightarrow \{v_n\}$ is relatively compakt in $W_2^k(\Omega)$

$\|v_n\|_{k+1} = 1$ $\qquad$ $\boxed{W_2^{k+1} \hookrightarrow W_2^k}$

$\Downarrow$ i.e.

$\exists \{v_{n'}\} \subset \{v_n\} \subset W_2^{k+1}(\Omega) \wedge \exists v \in W_2^k(\Omega)$:

$v_{n'} \xrightarrow[n' \to \infty]{} v \in W_2^k(\Omega)$ in $W_2^k(\Omega)$

Result:

$$\boxed{\begin{array}{l} v_{n'} \longrightarrow v \text{ in } W_2^k(\Omega) \\ \|\partial^\alpha v_{n'}\|_0 \longrightarrow 0, \ |\alpha| = k+1 \end{array}} \xRightarrow[\text{(mms) Def. 2.1}]{} \boxed{\begin{array}{l} v_{n'} \longrightarrow v \text{ in } W_2^{k+1}(\Omega) \\ \partial^\alpha v = 0 \ \forall \alpha : |\alpha| = k+1 \end{array}}$$

$\boxed{\operatorname{Ker} |\cdot|_{k+1} = P_k}$ (mms)

$\Longrightarrow$ $v = p_k \in P_k$ since $\partial^\alpha v = 0 \ \forall \alpha : |\alpha| = k+1$!

Furthermore, we have, for $i = 1,2,\dots,l$,

$0 \xleftarrow[n' \to \infty]{b)} f_i(v_{n'}) \xrightarrow[n' \to \infty]{} f_i(v) \overset{!}{=} 0$ since $v_{n'} \to v_n$ in $W_2^{k+1}(\Omega)$

$f_i(v_{n'}) \to 0, \ \forall i = \overline{1,l}$

$\Downarrow$

$\boxed{v = 0 \text{ since } v \in P_k}$

↯ $1 = \|v_{n'}\|_{k+1} = \|v\|_{k+1}$ since $v_{n'} \xrightarrow[n' \to \infty]{} v$ in $W_2^{k+1}(\Omega)$.

q.e.d.

Exercise 2.14: $\|\cdot\|^* \simeq \|\cdot\|$

Show that the following new norms $\|\cdot\|^*$ are equivalent to the standard norms $\|\cdot\|$ in the corresponding Sobolev spaces:

1. in $W_p^1(\Omega)$ with $\|\cdot\| := \|\cdot\|_{W_p^1(\Omega)}$:

   a) $\|u\|^*_{W_p^1(\Omega)} := \left(\left| \int_\Omega u(x)\,dx \right|^p + |u|^p_{W_p^1(\Omega)} \right)^{1/p}$

   (cf. the Poincaré inequality)

   b) $\|u\|^*_{W_p^1(\Omega)} := \left(\left| \int_{\partial\Omega} u(x)\,ds \right|^p + |u|^p_{W_p^1(\Omega)} \right)^{1/p}$

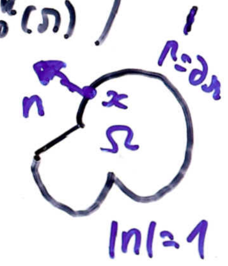
   c) $\|u\|^*_{W_p^1(\Omega)} := \left(\int_{\partial\Omega} |u(x)|^p\,ds + |u|^p_{W_p^1(\Omega)} \right)^{1/p}$

   ...

2. in $W_p^k(\Omega)$ with $\|\cdot\| := \|\cdot\|_{W_p^k(\Omega)}$:

   $$\|u\|^*_{W_p^k(\Omega)} := \left(\sum_{\ell=0}^{k-1} \int_{\partial\Omega} \left| \frac{\partial^\ell u}{\partial n^\ell} \right|^p ds + |u|^p_{W_p^k(\Omega)} \right)^{1/p},$$

   where $n = n(x) = (n_1, \dots, n_d)^T$ denotes the exterior unit normal at $x \in \Gamma = \partial\Omega$



3. in $\overset{\circ}{W}{}_p^k(\Omega)$ with $\|\cdot\| := \|\cdot\|_{W_p^k(\Omega)}$:

   Remark: Ass. $\partial\Omega \in C^{0,1}$ (Lip) is not needed!

   $\|u\|^*_{W_p^k(\Omega)} := |u|_{W_p^k(\Omega)}$,

   i.e. in the subspace $\overset{\circ}{W}{}_p^k(\Omega)$ of $W_p^k(\Omega)$, the standard semi-norm $|\cdot|_{W_p^k(\Omega)}$ is a norm that is equivalent to the norm $\|\cdot\|_{W_p^k(\Omega)}$!

Solution: see Tutorial 04

$\wedge$ 1a) on the BLACKBOARD!

1a) $\|u\|^*_{W^1_p(\Omega)} := \left(\left| \int_\Omega u(x)\,dx \right|^p + |u|^p_{W^1_p(\Omega)} \right)^{1/p} \simeq \|u\|_{W^1_p(\Omega)}$

$k=0,\ \ell=1,\ \sigma=1,\ s=k+\sigma=1$

$f_1(u) = \left| \int_\Omega u(x)\,dx \right| : W^1_p(\Omega) \longrightarrow \mathbb{R}^1_0$:

- semi-norm: trivial

Hölder: $\frac{1}{p}+\frac{1}{q}=1$

- $0 \le f_1(u) = \left| \int_\Omega 1 \cdot u\,dx \right| \le \left(\int_\Omega 1^q dx \right)^{1/q} \left(\int_\Omega |u|^p dx \right)^{1/p}$

$$= |\Omega|^{1/q} \|u\|_{L_p(\Omega)} \le |\Omega|^{1/q} \|u\|_{W^1_p(\Omega)}.$$

- $0 = f_1(u) = \left| \int_\Omega u\,dx \right| = |u|\,|\Omega| \Rightarrow u=0$

$u = c \in P_0$

Theorem 2.13 gives: $\exists\ \underline{c}, \overline{c} = \text{const} > 0$:

$$\underline{c} \left(\left| \int_\Omega u\,dx \right|^p + \underbrace{|u|^p_{W^1_p(\Omega)}}_{\overbrace{}^{=}} \right)^{1/p} \le$$

$$\le \left(\underline{\int_\Omega |u|^p dx} + \int_\Omega |\nabla u|^p dx \right)^{1/p} \le \underline{\overline{c} \left(\left| \int_\Omega u\,dx \right|^p + \int_\Omega |\nabla u|^p dx \right)^{1/p}}$$

q.e.d

that yield the famous <u>Poincaré inequality</u>

$$\boxed{\int_\Omega |u|^p dx \le \overline{c}^p \left[\left| \int_\Omega u\,dx \right|^p + \int_\Omega |\nabla u|^p dx \right]}$$

Since the proof of Theorem 2.13 is NOT constructive, we get no quantitative information about the Poincaré constant $c_P = \overline{c}^p$.

Friedrichs-type Inequalities:

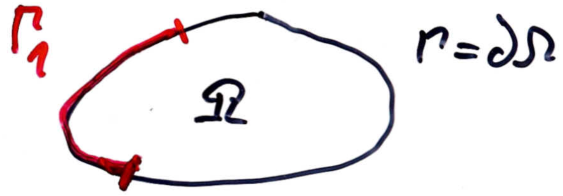
- Let us consider the following setting:

$\Gamma_1 \subset \Gamma = \partial\Omega$: $\mathrm{meas}_{d-1}(\Gamma_1) := \mathrm{meas}_{\mathbb{R}^{d-1}}(\Gamma_1) := \int_{\Gamma_1} ds > 0$,

$V_0 := \{ v \in W_p^1(\Omega) : \gamma_{\Gamma_1} v := v|_{\Gamma_1} = 0 \} \subset W_p^1(\Omega)$,

$V_0 = \overset{\circ}{W}{}_p^1(\Omega)$ if $\Gamma_1 = \Gamma$,

$1 \le p < \infty$



- **Lemma 2.15:**

Ass.: $1 \le p < \infty$; $\Gamma_1 \subset \Gamma$: $\mathrm{meas}_{d-1}(\Gamma_1) = |\Gamma_1| > 0$.

St.: Then there exists a positive constant $\bar{c} = \mathrm{const} > 0$ such that

$$(11) \quad \int_\Omega |u|^p dx \le \bar{c}^p \int_\Omega |\nabla u|^p dx \quad \forall u \in V_0 .$$

In the case $\Gamma_1 = \Gamma$, inequality (11) is also called Friedrichs' inequality: $c_F = \bar{c}$.

Proof: Using Sobolev's norm equivalence Theorem 2.13, we first show that

$$\|u\|^*_{W_p^1(\Omega)} := \left(f_1^p(u) + |u|^p_{W_p^1(\Omega)} \right)^{1/p} \simeq \|u\|_{W_p^1(\Omega)} \text{ in } W_p^1(\Omega)$$

with $f_1(u) := \left(\int_{\Gamma_1} |u|^p ds \right)^{1/p}$.

Indeed, $f_1(\cdot)$ fulfils the assumptions of Th. 2.13:

1) $f_1(\cdot) : W_p^1(\Omega) \to \mathbb{R}_0^1 = [0,\infty)$ is a semi-norm: (mms)

2) $0 \le f_1(u) = \left(\int_{\Gamma_1} |u|^p ds \right)^{1/p} \le \left(\int_\Gamma |u|^p ds \right)^{1/p}$

$$= \|u\|_{L_p(\Gamma)} \underset{(9)_0}{\le} c \, \|u\|_{W_p^1(\Omega)} .$$

3) $v \in \mathcal{P}_0$, i.e. $v = \text{const} \;\wedge\; 0 = f_1(v) = \left(\int_{\Gamma_1} |v|^p ds\right)^{1/p} = |v|\, |\Gamma_1|^{1/p}$

$\Rightarrow v \equiv 0$.

Th. 2.13 implies that $\exists\, \underline{c}, \overline{c} = \text{const} > 0$:

$$(10)\quad \underline{c}\left(\int_{\Gamma_1} |u|^p ds + \int_\Omega |\nabla u|^p dx\right)^{1/p} \le \left(\int_\Omega (|u|^p + |\nabla u|^p) dx\right)^{1/p} \le \overline{c}\left(\int_{\Gamma_1} |u|^p ds + \int_\Omega |\nabla u|^p dx\right)^{1/p}$$

$$\forall u \in W_p^1(\Omega)$$

For functions $u \in V_0 \subset W_p^1(\Omega)$, we get from (11)

$$(11)\quad \int_\Omega |u|^p dx \le \overline{c}^p \int_\Omega |\nabla u|^p dx \quad \forall u \in V_0.$$

q.e.d.

- **Corollary 2.16:**

The $W_p^1(\Omega)$ - semi-norm $|\cdot|_{W_p^1(\Omega)}$ is a norm on $V_0 = \{v \in W_p^1(\Omega) : \gamma_{\Gamma_1} u = 0\}$ which is equivalent to the standard $W_p^1(\Omega)$-norm $\|\cdot\|_{W_p^1(\Omega)}$:

$$(10)\quad \underline{c}\, |u|_{W_p^1(\Omega)} \le \|u\|_{W_p^1(\Omega)} \le \overline{c}\, |u|_{W_p^1(\Omega)} \quad \forall u \in V_0 \subset W_p^1(\Omega)$$

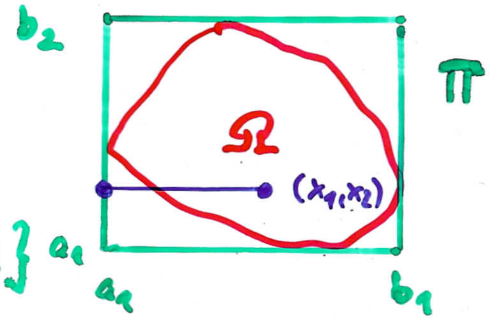
- **Exercise 2.17:** (Constr. Proof of Friedrichs' ineq.)

Show that $\exists\, c_F = \text{const} > 0$: $\quad (\Gamma_1 = \Gamma,\ p = 2)$

$$(11)\quad \int_\Omega (u(x))^2 dx \le c_F^2 \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in V_0 = \overset{\circ}{H}{}^1(\Omega)$$

with $c_F = \frac{1}{\sqrt{2}} \min_{i=1,d} (b_i - a_i)$,

where



$\Omega \subset \Pi := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d :$

$a_i < x_i < b_i,\ i = \overline{1,d}\}$

Hint:

$$u(x_1, x_2) = u(a_1, x_2) + \int_{a_1}^{x_1} \frac{\partial u}{\partial \xi_1}(\xi_1, x_2)\, d\xi_1.$$

■ Poincaré-type Inequalities:

• Lemma 2.16:

Ass.: $1 \le p < \infty$,
$\omega \subset \Omega$: $|\omega| := \operatorname{meas} \omega := \int_\omega dx > 0$, $* \wedge$ Lip

St.: Then there exists a constant $\bar{c} = \text{const} > 0$:

(12) $\int_\Omega |u|^p dx \le \bar{c}^p \left\{ \left| \int_\omega u(x) dx \right|^p + \int_\Omega |\nabla u|^p dx \right\} \; \forall u \in W_p^1(\Omega)$.

Proof:
Form Sobolev's norm equivalence Theorem 2,13, we get

$$\|u\|^*_{W_p^1(\Omega)} := \left(\left| \int_\omega u \, dx \right|^p + |u|^p_{W_p^1(\Omega)} \right)^{1/p} \simeq \|u\|_{W_p^1(\Omega)} .$$

Jndeed, $f_1(u) := \left| \int_\Omega u(x) \, dx \right|$ fulfils the assumption of Theorem 2.13:

1. $f_1(\cdot) : W_p^1(\Omega) \to \mathbb{R}^1_0$ is a semi-norm: (mms)
2. $0 \le f_1(u) = \left| \int_\omega 1 \cdot u \, dx \right| \underset{\uparrow}{\le} |\omega|^{1/q} \left(\int_\omega |u|^p dx \right)^{1/p}$

   Hölder

   $\le |\omega|^{1/q} \|u\|_{W_p^1(\Omega)} \quad \forall u \in W_p^1(\Omega)$.

3. $\left. \begin{array}{l} v = c \in P_0 \;\wedge \\ f_1(c) = \left| \int_\omega c \, dx \right| = |c| \cdot \operatorname{meas} \omega = 0 \end{array} \right\} \Rightarrow v = c = 0.$

q.e.d.

• In the case $\omega = \Omega$, inequality (12) is called Poincaré or Poincare-Friedrichs inequality (cf. Exercise 2.14. 1 a) !): $\bar{c} = c_p$

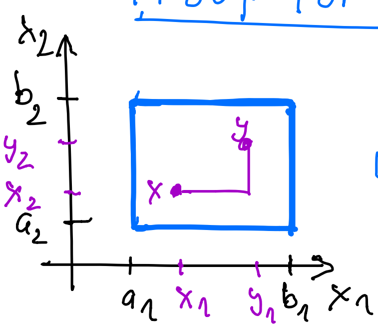
• Poincaré-Friedrichs inequality: $1 \le p < \infty$

$$(12)_{\omega=\Omega} \qquad \boxed{\int_\Omega |u|^p dx \le c_p^p \left\{ \left|\int_\Omega u\,dx\right|^p + \int_\Omega |\nabla u|^p dx \right\} \quad \forall u \in W_p^1(\Omega)}$$

Let us consider the most important case $p=2$:
If $\Omega = \Pi := \{x=(x_1,\dots,x_d) \in \mathbb{R}^d : a_i < x_i < b_i,\ i=\overline{1,d}\}$,
then we can give a constructive proof of Poincaré's inequality:

$$(12)_{\omega=\Pi} \qquad \boxed{\int_\Pi |u(x)|^2 dx \le \frac{1}{|\Pi|}\left(\int_\Pi u(x)\,dx\right)^2 + \max_{i\in\{1,\dots,d\}} (b_i-a_i)^2 \int_\Pi |\nabla u|^2 dx}$$

Proof for $d=2$:



$$u(y_1,y_2) - u(x_1,x_2) = \left[\int_{x_1}^{y_1} \frac{\partial u}{\partial \xi_1}(\xi_1,x_2)\,d\xi_1 + \int_{x_2}^{y_2} \frac{\partial u}{\partial \xi_2}(y_1,\xi_2)\,d\xi_2 \right]$$

$$\left(u(y_1,y_2) - u(x_1,x_2)\right)^2 = [\ -''-\ ]^2$$

$$\text{lhs} = \int_\Pi \int_\Pi \left(u^2(y) - 2u(y)u(x) + u^2(x)\right) dx\,dy = \int_\Pi \int_\Pi [\ -''-\ ]^2 dx\,dy$$

$$\text{lhs} = |\Pi| \int_\Pi u^2(y)\,dy - 2\int_\Pi u(y)\,dy \int_\Pi u(x)\,dx + |\Pi| \int_\Pi u^2(x)\,dx$$

$$= 2|\Pi| \int_\Pi u^2(x)\,dx - 2\left(\int_\Pi u(x)\,dx\right)^2$$

$$\int_\Pi u^2(x)\,dx \le \frac{1}{|\Pi|}\left(\int_\Pi u(x)\,dx\right)^2 + \frac{1}{2|\Pi|}\int_\Pi \int_\Pi [\ -''-\ ]^2 dx\,dy$$

$$\le \dots \ (\text{mms})$$

- **Equivalent forms of Poincaré's inequality:**

$$\boxed{\int_\Omega |u(x)|^2 dx \le \frac{1}{|\Omega|}\left(\int_\Omega u(x)dx\right)^2 + c_p^2 \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in W_2^1(\Omega)}$$

$$\Updownarrow$$

$$\boxed{\begin{aligned}\int_\Omega \Big|u(x) - \frac{1}{|\Omega|}\int_\Omega u(y)dy\Big|^2 dx &= \inf_{q \in P_0 = \mathbb{R}} \int_\Omega |u(x)-q|^2 dx \le \\ &\le c_p^2 \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in W_2^1(\Omega) = H^1(\Omega)\end{aligned}}$$

Proof:

$\Longleftarrow$
$$\inf_{q \in P_0} \int_\Omega |u(x)-q|^2 dx = \min_{q\in\mathbb{R}} \underbrace{\left\{ |\Omega| q^2 - 2\int_\Omega u(x)dx \cdot q + \int_\Omega u^2(x)dx \right\}}_{= f(q)}$$

$$f'(q) = 2|\Omega| q - 2\int_\Omega u(x)dx = 0 \Leftrightarrow q = \frac{1}{|\Omega|}\int_\Omega u(x)dx$$

i.e.
$$\inf_{q\in P_0} \int_\Omega |u(x)-q|^2 dx = \frac{1}{|\Omega|}\left(\int_\Omega u dx\right)^2 - 2\frac{1}{|\Omega|}\left(\int_\Omega u dx\right)^2 + \int_\Omega u^2 dx$$
$$\le c_p^2 \int_\Omega |\nabla u|^2 dx$$

$\Longrightarrow$ $u(x) = v(x) - q, \quad q \in P_0 = \mathbb{R}$

$$\int_\Omega |v(x)-q|^2 dx \le \frac{1}{|\Omega|}\left(\int_\Omega (v(x)-q)dx\right)^2 + c_p^2 \int_\Omega |\nabla v(x)|^2 dx$$

$$\inf_{q\in P_0} \int_\Omega |v(x)-q|^2 dx \le \frac{1}{|\Omega|} \underbrace{\min_{q\in\mathbb{R}}\left(\int_\Omega (v(x)-q)dx\right)^2}_{\ge 0 \ \wedge\ = 0 \text{ for } q = \frac{1}{|\Omega|}\int_\Omega v dx} + c_p^2 \int_\Omega |\nabla v|^2 dx$$

q.e.d.

- **Remark:** $p = 2$, Ω - $*$, Lip, convex: $c_p = \operatorname{diam}(\Omega)/\pi$.

Referenzen: L.E. Payne, H.F. Weinberger (1960): Arch. Rat. Mech. Anal.
M. Bebendorf (2003): Z. Anal. Anwendungen