2. Useful Tools from the Theory of Sobolev's Spaces [1) 2)]

2.1. Sobolev's Generalized Derivatives and Distributions

- Let $\Omega \subset \mathbb{R}^d$ be a bounded (*) Lip domain with the boundary $\Gamma = \partial\Omega$, $\Omega \neq \emptyset$, $d \in \mathbb{N}$ $(d = 1,2,3)$.
  Define
  $\mathring{C}^\infty(\Omega) :=$ space of all continuously infinite times differentiable real functions with compact support in Ω,
  and introduce the following <u>convergence</u> in $\mathring{C}^\infty(\Omega)$:

(1) $\boxed{\varphi_n \xrightarrow[n\to\infty]{} 0 \text{ in } \mathring{C}^\infty(\Omega) \text{ iff a) } \exists K \subset \Omega \text{ - compact}: \varphi_n(x) = 0 \;\forall x \notin K \;\forall n \in \mathbb{N}, \text{ b) } \partial^\alpha \varphi_n \underset{n\to\infty}{\rightrightarrows} 0 \text{ on } K \;\forall \alpha}$

where $\partial^\alpha \varphi = \dfrac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \alpha_1 + \ldots + \alpha_d$.

(2) $\varphi_n \xrightarrow{n\to\infty} \varphi$ iff $\varphi - \varphi_n \xrightarrow{n\to\infty} 0$ in $\mathring{C}^\infty(\Omega)$.

- $D(\Omega) := \mathring{C}^\infty(\Omega)$ equipped with the convergence (1)
  $=$ space of all fundamental functions

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1) To a great extend this Chapter should be a repetition from classes on PDEs or NuPDEs!

2) See also Chapter 3 of "Numerik I" that can be downloaded from the NuMa homepage:

Sobolev's Derivatives:

Def. 2.1:
A function $w \in L_{loc}(\Omega) :=$ {space of all locally integrable functions in Ω} is called αth Sobolev's generalized derivative of some function $u \in L_{loc}(\Omega)$ iff

(3) $$\int_\Omega u\, \partial^\alpha v\, dx = (-1)^{|\alpha|} \int_\Omega w \cdot v\, dx \quad \forall v \in \mathring{C}^\infty(\Omega).$$

Notation: $w = \partial^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$.

- **Properties of the generalized (weak) derivatives:**
  1. For $u \in C^K(\overline{\Omega})$, the classical and the generalized derivatives coincide up to the order K, i.e. $|\alpha| \le K$!
  2. The generalized derivatives are uniquely defined up to a set of measure zero!
  3. The generalized derivatives preserve almost all properties (e.g. calculus) of the classical derivatives!

Remark 2.2:
The concept of generalized (weak) derivatives can also be extended to other differential operators like
a) div of a vector function $u = (u_1, \ldots, u_d)^T \in [L_{loc}(\Omega)]^d$:

(4) $$w = \operatorname{div} u \in L_{loc}(\Omega): \int_\Omega u^T \nabla \varphi\, dx = -\int_\Omega w \varphi\, dx \quad \forall \varphi \in \mathring{C}^\infty(\Omega).$$

b) curl of a vector function $u = (u_1, \ldots, u_d)^T \in [L_{loc}(\Omega)]^d$:

(5) $$w = \operatorname{curl} u \in [L_{loc}(\Omega)]^d: \int_\Omega u \cdot \operatorname{curl} \varphi\, dx = \int_\Omega w \cdot \varphi\, dx \quad \forall \varphi \in [\mathring{C}^\infty(\Omega)]^d,$$

etc.

Distributions:

- **Def. 2.3:** Distributions = gen. functions ; $\mathcal{D}'(\Omega)$

Every linear and continuous functional on $\mathcal{D}(\Omega)$ is called distribution or generalized function.
The set of all distributions (= linear space) is denoted by $\mathcal{D}'(\Omega)$:

$$\langle \cdot , \cdot \rangle = \langle \cdot , \cdot \rangle_{\mathcal{D}' \times \mathcal{D}} : \mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \longrightarrow \mathbb{R}^1$$

i.e.

$$u \in \mathcal{D}'(\Omega) : \varphi \in \mathcal{D}(\Omega) \longrightarrow \langle u, \varphi \rangle \in \mathbb{R}^1 :$$

a) linear

b) continuous wrt the "convergence" introduced in $\mathcal{D}(\Omega)$:

$$\varphi_n \xrightarrow[n \to \infty]{} 0 \text{ in } \mathcal{D}(\Omega) \Rightarrow \langle u, \varphi_n \rangle \xrightarrow[n\to\infty]{} 0$$

- **Examples 2.4:**

1. Let $u \in L_{loc}(\Omega)$, e.g. $u \in L_p(\Omega) < L_{loc}(\Omega), p \geq 1$;
Then the relation

$$(6) \qquad \langle \tilde{u}, \varphi \rangle := \int_\Omega u(x) \varphi(x) \, dx \quad \forall \varphi \in \mathcal{D}(\Omega)$$

uniquely defines a distribution $\tilde{u} \in \mathcal{D}'(\Omega)$ (mms) that will be identified with $u \in L_{loc}(\Omega)$.
A distribution which has the representation (6) is called regular, otherwise singular.

2. Let $\xi \in \Omega$ be fixed. Then the relation

$$(7) \qquad \langle \delta_\xi, \varphi \rangle := \varphi(\xi) \quad \forall \varphi \in \mathcal{D}(\Omega)$$

obviously (mms) defines a distribution $\delta_\xi = \delta(\cdot - \xi) \in \mathcal{D}'(\Omega)$ called Dirac's δ-distribution. δ_ξ is a singular distr. (mms*)

- **Def. 2.5:** Distributive derivatives

Let $u \in D'(\Omega)$. Then the relation

$$(8) \qquad \langle u_\alpha, \varphi \rangle := (-1)^\alpha \langle u, \partial^\alpha \varphi \rangle \quad \forall \varphi \in D(\Omega)$$

obviously defines a distribution $u_\alpha \in D'(\Omega)$ for all $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$, $i = \overline{1,d}$ (mms).

We write

$$\partial^\alpha u = u_\alpha \in D'(\Omega),$$

and call $\partial^\alpha u$ αth distributive derivative of u.

A distribution obviously has distributive derivatives of arbitrary order!

- **Distributive derivatives vs Sobolev's derivatives:**

If the distributive derivative $\partial^\alpha u \in D'(\Omega)$ of a locally integrable function $u \in L_{loc}(\Omega)$ is regular, i.e. $\partial^\alpha u \in L_{loc}(\Omega)$, then there exists Sobolev's derivative in the sense of Def. 2.1 and both derivatives can be identified:

$$\langle \partial^\alpha u, \varphi \rangle := (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle \quad \forall \varphi \in D(\Omega)$$

$$\|$$

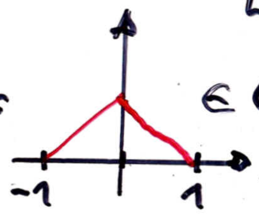
$$\int_\Omega \partial^\alpha u \, \varphi \, dx := (-1)^{|\alpha|} \int_\Omega u \, \partial^\alpha \varphi \, dx \quad \forall \varphi \in \overset{\circ}{C}{}^\infty(\Omega)$$

- **Example 2.6:**

$D'(\Omega)$
$\cup$
$L_{loc}(-1,1)$
$\cup$
$L_p(-1,1)$
$\cup$

Let us consider the function

$$u(x) = \begin{Bmatrix} 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le +1 \end{Bmatrix} = \quad \in C[-1,1]$$



a) $u' = \partial^1 u \in$?

b) $u'' = \partial^2 u \in$?

c) $u''' = \partial^3 u \in$?

...

2.2. The Sobolev Spaces $W_p^k(\Omega)$ and some Elementary Properties

■ <u>Repetition:</u> L_p-spaces (= Banach-spaces)

$L_p(\Omega) := \{u : \Omega \to \mathbb{R}^1 - \text{measurable} : \|u\|_{L_p(\Omega)} < \infty\}$,
where $1 \le p < \infty$ and
$$\|u\|_{L_p(\Omega)} := \Big(\int_\Omega |u|^p \, dx \Big)^{1/p} .$$

$L_\infty(\Omega) := \{u : \Omega \to \mathbb{R}^1 - \text{measurable} : \|u\|_{L_\infty(\Omega)} < \infty\}$,
where $\|u\|_{L_\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|$.

$L_2(\Omega)$ = Hilbert-space (H-space)
$$\|u\|_{L_2(\Omega)} = (u,u)^{1/2}_{L_2(\Omega)},$$
$$(u,v)_{L_2(\Omega)} = \int_\Omega u(x) v(x) \, dx$$

<u>Dual spaces:</u> $X^* = X'$

$[L_p(\Omega)]^* \equiv [L_p(\Omega)]' = L_q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$.

$[L_1(\Omega)]^* = L_\infty(\Omega)$

$[L_\infty(\Omega)]^* \neq L_1(\Omega)$, i.e. $L_1(\Omega)$ is not the dual space of $L_\infty(\Omega)$!

Sobolev-Spaces $W_p^k(\Omega)$, $1 \le p \le \infty$, $k = 0, 1, \dots$:

- **Def. 2.7:** $W_p^k(\Omega)$

$$W_p^k(\Omega) := \{ u \in L_p(\Omega) : \exists \partial^\alpha u \in L_p(\Omega) \ \forall \alpha : |\alpha| \le k \},$$

with the norm

$$\|u\|_{W_p^k(\Omega)} := \Big(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u(x)|^p dx \Big)^{1/p} \text{ for } p < \infty,$$

$$\|u\|_{W_\infty^k(\Omega)} := \max_{|\alpha| \le k} \|\partial^\alpha u\|_{L_\infty(\Omega)}.$$

Notation: $\partial^\alpha u = u$ for $|\alpha| = 0$, i.e. $\alpha = (0, \dots, 0)$,
$W_p^0(\Omega) = L_p(\Omega)$.

- Properties:

1. $W_p^k(\Omega)$ are B-spaces:
   - separable ($\exists$ countable dense subset),
   - uniformely convex for $1 < p < \infty$,
   - reflexive ($X = X^{**}$, $X = W_p^k(\Omega)$) for $1 < p < \infty$.
2. $H^k(\Omega) = W_2^k(\Omega)$ are H-spaces:
   $$\|u\|_{H^k(\Omega)} = \|u\|_k = (u, u)_{H^k(\Omega)}^{0.5} = (u, u)_k^{0.5},$$
   $$(u, v)_k := \sum_{|\alpha| \le k} \int_\Omega \partial^\alpha u \, \partial^\alpha v \, dx.$$
3. Define $\tilde{W}_p^k(\Omega) = H_p^k(\Omega) := \overline{C^\infty(\bar{\Omega})}^{\|\cdot\|_{W_p^k}}$ in $W_p^k(\Omega)$.
   Then
   $$\tilde{W}_p^k(\Omega) \underset{\neq}{\subseteq} W_p^k(\Omega) \,!$$
   in general

   If Ω is $\dagger$ and Lip, then
   $$\tilde{W}_p^k(\Omega) = W_p^k(\Omega) = \overline{C^\ell(\bar{\Omega})}^{\|\cdot\|_{W_p^k}}$$
   for $\ell \ge k$, e.g. $\ell = \infty$. Thus, in the following, we assume
   Ω is $\dagger$ and Lip
   Note that then we can use the "closure principle"!

- <u>Traces and trace spaces:</u>
  - Let $\Omega \subset \mathbb{R}^d$ be $\mp$ and Lip, $1 \le p < \infty$. The there exists a linear, continuous operator

$$(9)_0 \quad \begin{cases} \gamma_0 \in L(W_p^1(\Omega), L_p(\Gamma)): & \Gamma = \partial\Omega \\ \gamma_0 u(x) = u(x) \quad \forall x \in \Gamma = \partial\Omega \ \forall u \in C^1(\overline{\Omega}), \\ \|\gamma_0 u\|_{L_p(\Gamma)} \le c \|u\|_{W_p^1(\Omega)} \quad \forall u \in \overline{C^1(\Omega)}^{\|\cdot\|_{W_p^1}} = W_p^1(\Omega) \end{cases}$$

The function $g = \gamma_0 u$ is called generalized trace function of u on Γ.
γ_0 is called trace operator. The trace operator γ_0 as operator from $W_p^1(\Omega)$ to $L_p(\Gamma)$ is not surjective, i.e. not every function $g \in L_p(\Gamma)$ is a trace of some function $u \in W_p^1(\Omega)$!

  - Trace space for the space $H^1(\Omega)$:

$$H^{1/2}(\Gamma) := \gamma_0 H^1(\Omega) \subset L_2(\Gamma),$$

$$(9)_{1/2} \qquad \|g\|_{H^{1/2}(\Gamma)} := \inf_{\substack{u \in H^1(\Omega): \\ \gamma_0 u = g}} \|u\|_{H^1(\Omega)}$$

  - <u>Trace theorem:</u>

$$(9)_t \qquad \exists\, c_t = const > 0: \ \|\gamma_0 u\|_{H^{1/2}(\Gamma)} \le c_t \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)$$

  - <u>Inverse trace theorem:</u> (extension theorm)

$$(9)_e \qquad \exists\, c_e = const > 0: \ \forall g \in H^{1/2}(\Gamma)\ \exists u \in H^1(\Omega):$$

$$\gamma_0 u = g \text{ and}$$

$$\|u\|_{H^1(\Omega)} \le c_e \|g\|_{H^{1/2}(\Gamma)}$$

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The Sobolev-Spaces $\overset{\circ}{W}{}_p^k(\Omega)$, $1 \le p \le \infty$, $k = 0,1,\dots$:

- **Def. 2.8:**

$$\overset{\circ}{W}{}_p^k(\Omega) := \overline{\overset{\circ}{C}{}^\infty(\Omega)}^{\|\cdot\|_{W_p^k}} = \text{closure of } \overset{\circ}{C}{}^\infty(\Omega) \text{ in } W_p^k(\Omega).$$

Here we do not need the assumption that Ω is $\in$ Lip!

- **Properties of the spaces $\overset{\circ}{W}{}_p^k(\Omega)$:**

1. $\overset{\circ}{W}{}_p^k(\Omega)$ is a closed subspace of $W_p^k(\Omega)$ since $\overset{\circ}{C}{}^\infty(\Omega)$ is a linear subspace of $W_p^k(\Omega)$.
2. $\overset{\circ}{W}{}_p^0(\Omega) = L_p(\Omega)$ since $\overset{\circ}{C}{}^\infty(\Omega)$ is dense in $L_p(\Omega)$!
3. $\gamma_0 u =: u|_{\Gamma = \partial\Omega} = 0 \quad \forall u \in \overset{\circ}{W}{}_p^1(\Omega)$ (mms)
4. $\gamma_0 \partial^\alpha u = 0 \quad \forall u \in \overset{\circ}{W}{}_p^k(\Omega) \quad \forall \alpha : |\alpha| \le k-1$

$$\left(\Rightarrow \gamma_0 u = \gamma_0 \frac{\partial u}{\partial n} = \dots = \gamma_0 \frac{\partial^{k-1} u}{\partial n^{k-1}} = 0 \right)$$

5. $\overset{\circ}{H}{}^k(\Omega) = H_0^k(\Omega) = \overset{\circ}{W}{}_2^k(\Omega)$ is a H-space.

The spaces $W_q^{-k}(\Omega)$, $k = 0,1,2,\dots$:

- **Def. 2.9:**

Let $1 < p < \infty$, $\frac{1}{q} + \frac{1}{p} = 1$, $k = 0,1,2,\dots$

$$W_q^{-k}(\Omega) := \{ u \in D'(\Omega) : \|u\|_{W_q^{-k}(\Omega)} < \infty \},$$

with the so-called "negative" norm (mms)

$$\|u\|_{W_q^{-k}(\Omega)} := \sup_{\substack{\varphi \in D(\Omega) \\ \varphi \ne 0}} \frac{|\langle u, \varphi \rangle_{D' \times D}|}{\|\varphi\|_{W_p^k(\Omega)}}$$

- **Lemma 2.10:** $W_q^{-k}(\Omega) = [\overset{\circ}{W}{}_p^k(\Omega)]^*$

Ass.: Let $1 < p < \infty$, $q^{-1} + p^{-1} = 1$, $k = 0,1,2,\dots$;
St.: Then the spaces $W_q^{-k}(\Omega)$ and $[\overset{\circ}{W}{}_p^k(\Omega)]^*$ can be identified.

Proof: see Numerik I, pp. 67-68. ■

Two Special Function Spaces: → H-spaces

- $H(\mathrm{div}) = H(\mathrm{div},\Omega) := \{u \in [L_2(\Omega)]^d : \exists\, \mathrm{div}\, u \in L_2(\Omega)\}$
 = H-space with the scalar product

$$(u,v)_{H(\mathrm{div})} := (u,v)_{L_2(\Omega)} + (\mathrm{div}\, u, \mathrm{div}\, v)_{L_2(\Omega)},$$

$$\|u\|_{H(\mathrm{div})} := (u,u)^{1/2}_{H(\mathrm{div})}.$$

What about trace and inverse trace theorems ?

- $H(\mathrm{curl}) = H(\mathrm{curl},\Omega) := \{u \in [L_2(\Omega)]^d : \exists\, \mathrm{curl}\, u \in [L_2(\Omega)]^d\}$
 = H-space with the scalar product

$$(u,v)_{H(\mathrm{curl})} := (u,v)_{L_2(\Omega)} + (\mathrm{curl}\, u, \mathrm{curl}(v))_{L_2(\Omega)},$$

$$\|u\|_{H(\mathrm{curl})} := (u,u)^{1/2}_{H(\mathrm{curl})}.$$

What about trace and inverse trace theorems!

Sobolev-Slobodeckij-Spaces:

- **Def. 2.11:** $H^s(\Omega) = W_2^s(\Omega)$, $s \in \mathbb{R}$ (H-spaces)
 1) $s = k = \text{integer} \in \mathbb{Z}$: $H^s(\Omega) := W_2^s(\Omega)$ defined by Def. 2.7 ($s=k \geq 0$) and Def. 2.9 ($s=k<0$).
 2) $s > 0$: $s = k+\sigma$, $k \in \mathbb{N}_0$, $\sigma \in (0,1)$:
 $H^s(\Omega) := \{u \in H^k(\Omega) : |u|_{H^s(\Omega)} < \infty\}$ with
 $\|u\|^2_{H^s(\Omega)} = (u,u)_{H^s(\Omega)}$ (separable H-space)
 $(u,v)_{H^s(\Omega)} = (u,v)_s := (u,v)_{H^k(\Omega)} + (u,v)_{k+\sigma}$,
 $|u|^2_{H^s(\Omega)} := (u,u)_{k+\sigma}$

$$(u,v)_{k+\sigma} := \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{(\partial^\alpha u(x) - \partial^\alpha u(y))(\partial^\alpha v(x) - \partial^\alpha v(y))}{|x-y|^{d+2\sigma}}\, dx\, dy$$

 3) $s < 0$: $H^s(\Omega) := [\mathring{H}^{-s}(\Omega)]^*$, $\mathring{H}^{-s}(\Omega) = \overline{\mathring{C}^\infty(\Omega)}^{\|\cdot\|_{-s}}$

- Sobolev-Slobodeckij-spaces on manifolds, e.g. $H^s(\Gamma)$, $\Gamma = \partial\Omega$
- $H^{1/2}(\Gamma) \simeq \gamma_0 H^1(\Omega)$

- $H^{1/2}(\Gamma) \simeq \gamma_0 H^1(\Omega)$

$$H^{1/2}(\Gamma) := \gamma_0 H^1(\Omega) \simeq H^{1/2}(\Gamma) := \{g \in L_2(\Omega) : |g|_{H^{1/2}(\Gamma)} < \infty\}$$

$$\begin{array}{ccc} \Psi & & \Psi \\ g = \gamma_0 u & & g \end{array}$$

$$\|g\|_{H^{1/2}(\Gamma)} := \inf_{\substack{u \in H^1(\Omega) \\ \gamma_0 u = g}} \|u\|_{H^1(\Omega)} \simeq \|g\|_{H^{1/2}(\Gamma)} := \left(\|g\|^2_{L_2(\Gamma)} + |g|^2_{H^{1/2}(\Gamma)}\right)^{1/2}$$

norm equivalence (see Def. 2.12)

with the $H^{1/2}$-semi-norm

$$|g|^2_{H^{1/2}(\Gamma)} := \int_\Gamma \int_\Gamma \frac{|g(x) - g(y)|^2}{|x-y|^{(d-1)+1}}\, dy\, dx$$

- **Def. 2.12:** $\|\cdot\|_{(1)} \simeq \|\cdot\|_2$

Two norms $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ defined on some Linear space X are called equivalent iff $\exists$ positive, fixed constantant $\underline{c}$ and $\bar{c}$ such that

$$\underline{c}\, \|u\|_{(2)} \le \|u\|_{(1)} \le \bar{c}\, \|u\|_{(2)} \quad \forall u \in X.$$