b 05-01

1.2.4. H(curl) - Elliptic Problems: Electromagnetic Fields

■ Crash course on "Electromagnetics (EM)"

see Chapter 4 of the lectures on "Mathematical Models in Engineering" that were in the WS 2017/18 at JKU:

http://www.numa.uni-linz.ac.at/Teaching/LVA/2017w/MathModTech

Slides 34 - 46:

4.1. The Full MAXWELL Equations
→ Slides 34-40

4.2. Special Electromagnetic Regimes
→ Slides 41-45

Appendix: Vector, Differential and Integral Identities
→ Slide 46

See also Bachelor Thesis by Bernhard Oberndorfer
~/Teaching/Bachelor

■ **Result is always:** curl-curl - problem, cf. also $(20)_{CF}$:

$$\boxed{\text{Find } u=(u_1,u_2,u_3)^T: \quad \text{curl}\left(\frac{1}{\mu}\,\text{curl}(u)\right) + \underset{\mathbb{C}}{\underset{\Uparrow}{\varkappa}}\, u = f \text{ in } \Omega \quad + BC}$$

$\varkappa = 0$ - magnetostatic

$\mathbb{R}^+ \ni \varkappa = \sigma$ - conductivity reg.

$\mathbb{R}^+ \ni \varkappa = \frac{\varepsilon}{\tau^2} + \frac{\sigma}{\tau}$ - impl. time discretization

$\mathbb{C} \ni \varkappa = i\omega\sigma - \varepsilon\omega^2$ - time-harmonic case

see also Slide 45

The space $H(curl)$ and $H_0(curl)$: → Ch. 2

The definition of $curl(u)$:

a) Classical definition:

If $u = (u_1, u_2, u_3)^T \in [C^1(\bar{\Omega})]^3$, then

$$curl(u) = rot(u) = \nabla \times u$$

$$:= \det \begin{bmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{bmatrix} \in [C(\bar{\Omega})]^3$$

b) Generalized (weak) definition:

The vector-function $w = (w_1, w_2, w_3)^T \in [L_1(\Omega)]^3$ is called $curl(u)$ of a vector-function $u = (u_1, u_2, u_3)^T \in [L_1(\Omega)]^3$ if the following integral identity holds:

$$\int_\Omega w \cdot \varphi \, dx = \int_\Omega u \cdot curl(\varphi) \, dx \quad \forall \varphi \in [\overset{\circ}{C}{}^\infty(\Omega)]^3.$$

Notation: $w = curl(u) \in [L_1(\Omega)]^3$.

$H(curl) := \{ u = (u_1, u_2, u_3)^T \in [L_2(\Omega)]^3 : curl(u) \in [L_2(\Omega)]^3 \}$

$$= \overline{C^\infty(\bar{\Omega})^3}^{\|\cdot\|_{H(curl)}} = H(curl, \Omega)$$

$\uparrow$ Ω = Lip $\wedge$ $\not{\ }$

is a Hilbert-space; $\|u\|^2_{H(curl)} = (u, u)_{H(curl)}$,

$$(u, v)_{H(curl)} := (u, v)_{L_2(\Omega)} + (curl(u), curl(v))_{L_2(\Omega)}$$

- Integration by parts gives (mms: in the smooth case)

$$\int_\Omega \operatorname{curl}(u)\cdot\varphi\,dx = \int_\Omega u\cdot\operatorname{curl}(\varphi)\,dx + \langle \gamma_\tau u, \varphi\rangle$$

$$\langle \gamma_\tau u, \varphi\rangle = \int_\Omega (u\times n)\cdot\varphi\,ds$$

$$\forall u \in H(\operatorname{curl}) = \overline{C^\infty(\overline{\Omega})^3}^{H(\operatorname{curl})} \quad \text{closure}$$

$$\forall \varphi \in H^1(\Omega)^3 = \overline{C^\infty(\overline{\Omega})^3}^{H^1(\Omega)}$$

with the trace operator

$$\gamma_\tau : H(\operatorname{curl}) \longrightarrow \operatorname{im}\gamma_\tau := H_{\parallel}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \subset H^{-1/2}(\Gamma)^3$$

$$\gamma_\tau u := u\times n|_\Gamma$$

$$\|\gamma_\tau u\|_{H^{-1/2}(\Gamma)} \leq c\,\|u\|_{H(\operatorname{curl})} \quad \forall u\in H(\operatorname{curl})$$

- $H_0(\operatorname{curl}) := \{u \in H(\operatorname{curl}) : \gamma_\tau u := u\times n|_\Gamma = 0\}$

- $\operatorname{Ker}(\operatorname{curl}) = \nabla H^1(\Omega)$

- **Helmholtz-decomposition:**

$\forall u \in L_2(\Omega)^3\ \exists \varphi\in H^1(\Omega)\ \exists \psi \in H(\operatorname{curl})$:

$$u = \nabla\varphi \overset{L_2}{\oplus} \operatorname{curl}\psi$$

provided that Ω is $\dagger$, Lip, and simply connected!

Variational Formulation of the curl-curl-Problem:

- Let us consider the model curl-curl-problem:

$$(20)_{CF} \quad \boxed{\begin{array}{l} \text{Find vector-function } u=(u_1,u_2,u_3)^T: \\ \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}(u)\right)+\varkappa u = f \text{ in } \Omega \\ \qquad\qquad u\times n = 0 \text{ on } \Gamma \end{array}}$$

with given $\mu(x)$, $\varkappa(x)$, and $f=(f_1,f_2,f_3)$.

For simplicity, we assume that $\varkappa$ is a real fuction.

- **Derivation of the Variational Formulation:**

① $V_0 = H_0(\operatorname{curl}) = \{ v\in V = H(\operatorname{curl}) : v\times n = 0 \text{ on } \Gamma \}$,

② $\int_\Omega \left[\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}(u)\right)\cdot v + \varkappa\, u\cdot v\right]dx = \int_\Omega f\cdot v\,dx \quad \forall v\in V_0$

③ $\int_\Omega \left[\frac{1}{\mu}\operatorname{curl}(u)\cdot\operatorname{curl}(v) + \varkappa u v\right]dx + \int_\Gamma \left(\underbrace{\tfrac{1}{\mu}\operatorname{curl}(u)\times n}\right) v\,ds = \int_\Omega f v\,ds$

$= \mu^{-1}\operatorname{curl}(u)\cdot(v\times n) = \mu^{-1}\operatorname{curl}(u)\times n\cdot(v\times n)$

④ $\int_\Gamma \underbrace{\frac{1}{\mu}\operatorname{curl}(u)}\cdot\underbrace{(v\times n)}_{=0} ds = 0 \quad \forall v\in V_0 = H_0(\operatorname{curl})$

$\longrightarrow$ essential BC: $u\times n = g_1$ on Γ_1

$\longrightarrow$ natural BC: $\mu^{-1}\operatorname{curl}(u)\times n = H\times n = g_2$ on Γ_2

⑤ $V_g = \{ v\in V = H(\operatorname{curl}) : v\times n = g_1 := 0\} = V_0$

- Therefore, we arrive at the following VF:

$$(20)_{VF} \quad \boxed{\begin{array}{l} \text{Find } v \in \tilde{V}_g = \tilde{V}_0 = H_0(curl): \\ \underbrace{\int_\Omega \left[\frac{1}{\mu} curl(u) \cdot curl(v) + \varkappa u \cdot v\right] dx}_{= a(u,v)} = \underbrace{\int_\Omega f \cdot v dx}_{= \langle F, v \rangle} \quad \forall v \in \tilde{V}_0 \end{array}}$$

- Lax-Milgram's Theorem yields (mms):

If $\mu, \varkappa \in L_\infty(\Omega)$: $\exists \underline{\mu}, \overline{\mu}, \underline{\varkappa}, \overline{\varkappa} = const > 0$:

$0 < \underline{\mu} \le \mu(x) \le \overline{\mu}$ and $0 < \underline{\varkappa} \le \varkappa(x) \le \overline{\varkappa}$ $\overset{o}{\forall} x \in \Omega$,

then there exists a unique solution $u \in \tilde{V}_0$ of $(20)_{VF}$.

- The magnetostatic case ($\varkappa = 0$):

1. In the magnetostatic case ($\varkappa = 0$) with permeability $\mu \in L_\infty(\Omega)$: $0 < \underline{\mu} \le \mu(x) \le \overline{\mu}$ $\overset{o}{\forall} x \in \Omega$, the solution $u \in \tilde{V}_0$ of $(20)_{VF}$ is only unique up to gradient fields $\nabla \varphi$, $\varphi \in \overset{o}{H}{}^1(\Omega)$; i.e., however, that $B = curl(A) = curl(u) = curl(u + \varphi)$ is unique!
In order to prove existence ($\exists$), the right-hand side f must be weakly divergence-free, i.e.

$$\underbrace{\int_\Omega f \cdot \nabla \varphi \, dx}_{\overset{f \in H(div)}{\longrightarrow} \; = \; -\int_\Omega div f \cdot \varphi \, dx} = 0 \quad \forall \varphi \in \overset{o}{H}{}^1(\Omega)$$

$$\boxed{\nabla_0 \; v = \nabla \varphi \in H_0(curl)}$$

2. If $\varkappa \in L_\infty(\Omega)$: $0 \le \varkappa(x) \le \overline{\varkappa}$ $\forall x \in \Omega$,
then the so-called "conductivity regularization"

$$\varkappa_\varepsilon = \max\{\varkappa, \varepsilon\}, \quad \varepsilon > 0$$

yields a unique solution $u_\varepsilon \in \mathbb{V}_0$ that converges to $u \in \mathbb{V}_0$ for $\varepsilon \to 0$, i.e.

$$\underset{(20)_{VF,\varkappa_\varepsilon}}{u_\varepsilon} \longrightarrow \underset{(20)_{VF}}{u} \text{ in } \mathbb{V} = H(\mathrm{curl}) \text{ for } \varepsilon \to 0 \quad O(\varepsilon)$$

3. Mixed Formulation see Tutorial 4!

Find $u \in \mathbb{V}_0 = H_0(\mathrm{curl})$ and $p \in \overset{\circ}{H}{}^1(\Omega)$:

$$\int_\Omega \frac{1}{\mu} \mathrm{curl}(u) \cdot \mathrm{curl}(v)\,dx + \int_\Omega v \cdot \nabla p\,dx = \int_\Omega f \cdot v\,dx \quad \forall v \in \mathbb{V}_0$$

$$\int_\Omega u \cdot \nabla q\,dx = 0 \quad \forall q \in \overset{\circ}{H}{}^1(\Omega)$$

If $\int_\Omega f \cdot \nabla \varphi\,dx = 0 \;\forall \varphi \in \overset{\circ}{H}{}^1(\Omega)$, then this mixed variational problem has a unique solution $(u, p) \in H_0(\mathrm{curl}) \times \overset{\circ}{H}{}^1(\Omega)$ and $p = 0$!

1.3. Other Variational Formulations

1.3.1. Mixed Variational Formulations

- Mixed variational formulations of the form

(21)
$$\boxed{\begin{array}{l}\text{Find } u \in V \text{ and } p \in Q \text{ such that}\\ a(u,v) + b(v,p) = \langle F, v\rangle \quad \forall v \in V,\\ b(u,q) - c(p,q) = \langle G, q\rangle \quad \forall q \in Q,\\ \text{where } V, Q \text{ - Hilbert spaces, } F \in V^*, G \in Q^* \text{ given,}\\ \quad \langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}^1 \text{ - duality product,}\\ \quad \langle \cdot, \cdot \rangle_Q : Q^* \times Q \to \mathbb{R}^1 \text{ - duality product,}\\ \quad a(\cdot,\cdot) : V \times V \to \mathbb{R}^1 \text{ - non-negative, cont. bilinear f.}\\ \quad b(\cdot,\cdot) : V \times Q \to \mathbb{R}^1 \text{ - continuous bilinear form,}\\ \quad c(\cdot,\cdot) : Q \times Q \to \mathbb{R}^1 \text{ - non-negative, cont. bilinear f.}\end{array}}$$

will be considered in the lectures "Computational "CM" = Methods in Continuum Mechanics" (→ Zulehner), and naturally arise in Fluid Mechanics

1) Oseen problem,
2) Stokes problem,

or artificially by introducing some dual variable

3) Hellinger-Reisner formulation of the Poisson equation or of the Linear 3D elasticity problem,
4) Mixed formulations of the plate bending problem.

■ Fluid Mechanics: (→ Lectures "MathMod")

0) Stationary Navier-Stokes Equations:

describing the stationary flow of an incompressible Newtonian fluid ($d=3$):

(22)$_{CF}$

Find the velocity field $u(x) = (u_1(x), u_2(x), u_3(x))^T$ and the pressure field $p(x)$:

$$-\frac{1}{Re}\Delta u + \underbrace{(u\cdot\nabla)u}_{\text{non-Linear convection term}} + \nabla p = f \text{ in } \Omega \subset \mathbb{R}^3,$$

$$\operatorname{div} u = 0 \text{ in } \Omega,$$

+ BC: e.g. $u=0$ on $\Gamma = \partial\Omega$

VF ⇓ (1) – (5)

$Re = \frac{\varrho}{\mu} \ell^* v^* = \frac{1}{\nu} \ell^* v^*$ – dimensionless Reynolds number

(22)$_{VF}$

Find $u \in V := (\overset{\circ}{H}{}^1(\Omega))^3$ and $p \in Q := \{q \in L_2(\Omega) : \int q \cdot 1\,dx = 0\}$:

$$a(u;u,v) + b(v,p) = \langle F, v\rangle \quad \forall v \in V$$

$$b(u,q) = \langle G, q\rangle \quad \forall q \in Q$$

where $a(\cdot;\cdot,\cdot): V\times V\times V \to \mathbb{R}^1$ – cont. trilinear form,

$$a(w;u,v) := \frac{1}{Re}\int_\Omega \nabla^T u\cdot\nabla v\,dx + \int_\Omega \sum_{i=1}^{3}\sum_{j=1}^{3} \underset{L_4}{w_j}\, \underset{L_2}{\frac{\partial u_i}{\partial x_j}}\, \underset{L_4}{v_i}\,dx, \quad \text{OK}$$

$b(u,q) := \int_\Omega \operatorname{div} u \cdot q\,dx$ – continuous bilinear form,

$\langle F, v\rangle := \int_\Omega f^T v\,dx, \quad G = 0$

Due to the convection term the N-S-problem (22) is non-Linear. Solvability investigation ($\exists + !$) is more difficult! The fix-point linearization leads to the so-called Ossen problem!

■ The OSEEN problem in variational formulation:

(23)
$$\boxed{\begin{array}{lll} \text{Find } u \in V \text{ and } p \in Q: & & \\ a(w;u,v) + b(v,p) & = \langle F,v\rangle & \forall v \in V, \\ b(u,q) & = 0 & \forall q \in Q. \end{array}}$$

For given $\overset{\overset{V}{\in}}{w}$, set $a(u,v) := a(w;u,v)$.
Then (23) exactly has the form (21), where the bilinear form $a(\cdot,\cdot)$ is here non-symmetric!

■ The STOKES problem:

For small Reynolds numbers (viscous flow), the (non-symmetric) convection term can be neglected. Thus, we obtain the VF:

(24)
$$\boxed{\begin{array}{lll} \text{Find } u \in V \text{ and } p \in Q: & & \\ a(u,v) + b(v,p) & = \langle F,v\rangle & \forall v \in V \\ b(u,q) & = 0 & \forall q \in Q \end{array}}$$

with the symmetric bilinear form
$$a(u,v) := \frac{1}{Re} \int_\Omega \nabla^T u \cdot \nabla v \, dx.$$

In the Lect. "CM", the existence and uniqueness of the solution $(u,p) \in V \times Q$ will be shown, i.e. the pressure p is unique only up to an additive constant, and the FE discretization will be discussed.

$$\downarrow$$

$$\underbrace{V_h \subset V, \; Q_h}_{\text{discrete LBB condition!}} \subset V \;\leadsto\; \text{mixed FEM}$$

Solid Mechanics:

3) Dirichlet problem for Poisson's eqn ($\hat{=}$ elasticity)

- Idea: Hellinger – Reisner

(25)
$$\boxed{\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma \\ \Omega &\subset \mathbb{R}^d \ * \end{aligned}} \longrightarrow \boxed{\begin{aligned} \sigma - \nabla u &= 0 \text{ in } \Omega \\ \operatorname{div} \sigma &= -f \text{ in } \Omega \\ u = 0 &\text{ on } \Gamma = \partial\Omega \end{aligned}} \quad \begin{aligned} &\int_\Omega \cdot\, \tau \, dx \ \forall \tau \in V \\ &\int_\Omega \cdot\, q \, dx \ \forall q \in Q \end{aligned}$$

← nat. BC !!!

- $\int_\Omega \sigma^T \tau \, dx + \int_\Omega u \cdot \operatorname{div} \tau - \int_\Gamma u \cdot \tau^T n \, ds = 0 \ \forall \tau \in V,$
- $\int_\Omega \operatorname{div} \sigma \cdot q \, dx = -\int_\Omega f \cdot q \, dx \ \forall q \in Q.$

- Spaces: $V = H(\operatorname{div}, \Omega) := \{\tau \in [L_2(\Omega)]^d : \operatorname{div} \tau \in L_2(\Omega)\},$

  with $\|\tau\|_V^2 := \sum_{i=1}^d \|\tau_i\|_{L_2(\Omega)}^2 + \|\operatorname{div} \tau\|_{L_2(\Omega)}^2,$

  $w = \operatorname{div} \tau \in L_2(\Omega)$:

  $\int_\Omega \tau^T \nabla \varphi \, dx = -\int_\Omega w \cdot \varphi \, dx \ \forall \varphi \in \overset{\circ}{H}{}^1(\Omega)$

  $Q = L_2(\Omega)$ with $\|q\|_Q := \|q\|_{L_2(\Omega)}$.

- Mixed Variational Formulation: $f \in L_2(\Omega)$ given

$(25)_{MVF}$

Find $\sigma \in V$ and $u \in Q$:

$$\begin{aligned} \int_\Omega \sigma^T \tau \, dx + \int_\Omega u \cdot \operatorname{div} \tau \, dx &= 0 \quad \forall \tau \in V \\ a(\sigma, \tau) + b(\tau, u) &= \langle F, \tau \rangle \\ \int_\Omega q \cdot \operatorname{div} \sigma \, dx &= -\int_\Omega f \cdot q \, dx \ \forall q \in Q \\ b(\sigma, q) &= \langle G, q \rangle \end{aligned}$$

- Advantages! ⊕ $u = 0$ on Γ is now a natural BC!
  ⊕ $\sigma = \nabla u$ (stresses) are often more interesting than u
  ⊕ later: Galerkin $V_h \subset V$, $Q_h \subset Q$ ↝ mixed FEM!
- Disadvantages! ⊖ scalar PDE of 2nd order
  ⟹ system of (d+1) PDEs of 1st order!
  ⊖ discrete problems are indefinite!

■ Mixed VF of the 1st biharmonic BVP:

- **Idea:** CIARLET–RAVIAR (1974), $\Omega \subset \mathbb{R}^2$, $\Gamma = \partial\Omega$

$$(26)\quad \boxed{\begin{array}{l} \Delta^2 u = f \text{ in } \Omega \\ u = 0 \text{ on } \Gamma \\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma \end{array}} \longrightarrow \boxed{\begin{array}{l} w - \Delta u = 0 \text{ in } \Omega \\ \Delta w = -f \text{ in } \Omega \\ u = 0 \text{ on } \Gamma \\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma \end{array}} \quad \begin{array}{l} \int_\Omega \cdot\, v\,dx \;\forall v \in V \\ \int_\Omega \cdot\, q\,dx \;\forall q \in Q \end{array}$$

nat. BC!

$V = H^1(\Omega)$
$Q = \mathring{H}^1(\Omega)$

- $\int_\Omega w \cdot v\,dx + \int_\Omega \nabla^T u \nabla v\,dx - \int_\Gamma \frac{\partial u}{\partial n} v\,ds = 0 \quad \forall v \in V$
- $\int_\Omega \nabla^T w \cdot \nabla q\,dx - \int_\Gamma \frac{\partial w}{\partial n} \underbrace{q}_{=0}\,ds = -\int_\Omega f q\,dx \quad \forall q \in Q$

- **Mixed Variational Formulation:** $f \in L_2(\Omega)$ given

$(26)_{MVF}$

Find $w \in V = H^1(\Omega)$ and $u \in Q = \mathring{H}^1(\Omega)$:

$$\underbrace{\int_\Omega w \cdot v\,dx}_{a(w,v)} + \underbrace{\int_\Omega \nabla^T u \cdot \nabla v\,dx}_{b(v,u)} = \underbrace{0}_{=\langle F, v\rangle_V} \quad \forall v \in V,$$

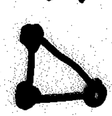
$$\underbrace{\int_\Omega \nabla^T w \cdot \nabla q\,dx}_{b(w,q)} = \underbrace{-\int_\Omega f \cdot q\,dx}_{=\langle G, q\rangle_Q} \quad \forall q \in Q,$$

- **Advantages / Disadvantages:**

⊕ Scalar 4th order PDE ⇒ System of 2 PDEs of 2nd order
→ i.e. only 1st order derivatives!

⊕ $\frac{\partial u}{\partial n}$ becomes natural BC!

⊕ Function $w = \Delta u$ ($\triangleq$ bending moment in the plate bending, or vorticity in fluid mechanics) is directly computed!

⊕ FE-Galerkin-discretization with C^0 elements, e.g. 

⊖ Discretized problem is indefinit!

1.3.2. Dual Variational Formulations

- Let us first consider the minimum problem

$(27)_{MP}$ Find $u \in V_g$: $E(u) = \inf_{v \in V_g} E(v)$

with the primal energy functional (Ritz-functional)

(28) $E(v) := \frac{1}{2} a(v,v) - \langle F, v \rangle$,

and let us assume that

- $V_g = g + V_0 = \{ u \in V : u = g + v, v \in V_0 \}$ = linear manifold = hyperplane
- $V_0 \subset V$ – closed subspace of the Hilbert-space $V, \|\cdot\|, (\cdot,\cdot)$,
- Standard assumption (→ LAX-MILGRAM: 1), 2), 2a), 2b)),
- $a(\cdot,\cdot)$ is symmetric on V_0.

Then the MP $(27)_{MP}$ is equivalent to the VP

$(27)_{VP}$ Find $u \in V_g$: $a(u,v) = \langle F, v \rangle \;\; \forall v \in V_0$,

and, thanks to the LAX-MILGRAM-theorem, it has a unique solution $u \in V_g$.

- The equivalent to $(27)_{MP} \equiv (27)_{VP}$ <u>dual VP</u> can be formulated as follows:

$$(27)_{DVP} \quad \text{Find } w \in W : G(w) = \sup_{v \in W} G(v)$$

with the dual (complementary) energy functional (Trefftz)

$$(28) \quad G(v) = -\tfrac{1}{2} a(v,v) + a(g,v) - \langle F, g \rangle$$

and the linear manifold W of all solutions of $(27)_{VP}$ in $\tilde{V}$, i.e.

$$W = \{ w = \bar{w} + v_0 : v_0 \in \tilde{U}_0 \} = \{ w \in \tilde{V} : a(w,v) = \langle F, v \rangle \ \forall v \in \tilde{V}_0 \}$$

$$\hookrightarrow \tilde{U}_0 = \{ v_0 \in \tilde{V} : a(v_0, v) = 0 \ \forall v \in \tilde{V}_0 \}$$

$$\hookrightarrow \bar{w} \in \tilde{V} : a(\bar{w}, v) = \langle F, v \rangle \ \forall v \in \tilde{V}_0$$

- Then we obtain by simple calculations (mws)

$$2 \| v - w \|^2 = E(v) - G(w) \ \forall v \in \tilde{V}_g \ \forall w \in W$$

$$2 \| v - u_* \|^2 = E(v) - E(u_*) \ \forall v \in \tilde{V}_g$$

$$2 \| w - u_* \|^2 = E(u_*) - G(w) \ \forall w \in W,$$

$$W \cap V_g = \{ u_* \},$$

where $\| \cdot \|^2 := a(\cdot, \cdot)$ denotes the energy norm. Therefore, the MAXIMUM and the MINIMUM of the TREFFTS- and the RITZ-functionals will be realized exactly for $u = w = u_*$, and the strong duality principle is valid, i.e.

$$(29) \quad \max_{v \in W} G(v) = G(u_*) = E(u_*) = \min_{v \in \tilde{V}_g} E(v),$$

- **Let us now consider the DIRICHLET problem for the POISSON equation as an EXAMPLE:**

$(30)_{CF}$ $\quad -\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\Gamma = \partial\Omega$

$\Rightarrow V = H^1(\Omega)$, $V_g = V_0 = \overset{\circ}{H}{}^1(\Omega)$ (since $g = 0$)

$(30)_{MP}$ Find $u \in V_0$: $E(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v \, dx \to \min!$

$\Updownarrow$

$(30)_{VP}$ Find $u \in V_0$: $\int_\Omega \nabla^T u \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in V_0$

$\Updownarrow$

$(30)_{DVP}$ Find $w \in W$: $G(w) := -\frac{1}{2}\int_\Omega |\nabla w|^2 dx \to \max!$

$W := \{ w \in H^1(\Omega) : \int_\Omega \nabla^T w \nabla v \, dx = \int_\Omega f v \, dx \ \forall v \in V_0 \}$

$= \{ w \in H^1(\Omega) : -\operatorname{div} \nabla w = f$ in the weak sense$\}$

! no BC !

$\Updownarrow \quad \boxed{\sigma = \nabla u}$

$(30)_{DP}$ Find $\sigma \in W(f)$: $G(\sigma) = \sup_{\tau \in W(f)} G(\tau)$

with $G(\tau) := -\frac{1}{2}\int_\Omega |\tau|^2 dx$,

$W(f) := \{ \tau \in H(\operatorname{div}, \Omega) : -\operatorname{div} \tau = f$ in $\Omega \}$

Remarks:

1. $(30)_{MP/VP}$: V_0 $(= V_g)$ $\rightsquigarrow$ essential BC must be fulfilled!

   $(30)_{DVP/DP}$: W or $W(f)$ $\rightsquigarrow$ PDE must be fulfilled in a weak sense!

2. primal $\longrightarrow$ mixed $\longleftarrow$ dual

   Lagrange-functional

   $L(\tau, q) = \frac{1}{2}\int_\Omega |\tau|^2 dx + \int_\Omega q \, (\operatorname{div} \tau + f) \, dx$