1.2.2. BVP for 2nd-Order Elliptic PDE Systems: The Linear Elasticity Problem

- **Classical Formulation of the Linear 3d elasticity problem** (see Lectures on MathModTech, Sec. 2.2)

(8) Find displacement field $u(x) = (u_1(x), u_2(x), u_3(x))^T \in X$:

1) equilibrium of forces: $-\operatorname{div} \sigma = f$

$$-\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \underset{\underset{\sigma_{ij}}{\parallel}}{\sigma_{ji}}(u(x)) = f_i(x) \quad \forall x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \; i = 1,2,3 \quad (*)$$

Kinetics

$\Leftarrow$ equilibrium of momentum: $\sigma = \sigma^T$

2) Material Laws = Hook's Law: $\sigma = D\varepsilon$

$$\sigma_{ij} = \sum_{k,l=1}^{3} D_{ijkl} \varepsilon_{kl} = \lambda \Big(\sum_{k=1}^{3} \varepsilon_{kk}\Big) \delta_{ij} + 2\mu \varepsilon_{ij}$$

21 independent elast. coeff. of 81

isotropic material:
$D_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
with Lamé's const $\lambda, \mu = \text{const} > 0$,
where $\lambda \neq \lambda(x)$, $\mu \neq \mu(x)$ for hom. mat.

3) Geometrical Strain-Displacement rel.: $\varepsilon = \frac{1}{2}(\nabla u + \nabla^T u)$

Kinematics

$$\varepsilon_{ij} = \varepsilon_{ij}(u) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ji} \quad \forall i,j = \overline{1,3}$$

4) Boundary Conditions:

1st Kind: $u(x) = 0$ (or $u(x) = \bar{u}(x)$) $\forall x \in \Gamma_1 = \Gamma_D = \Gamma_u$

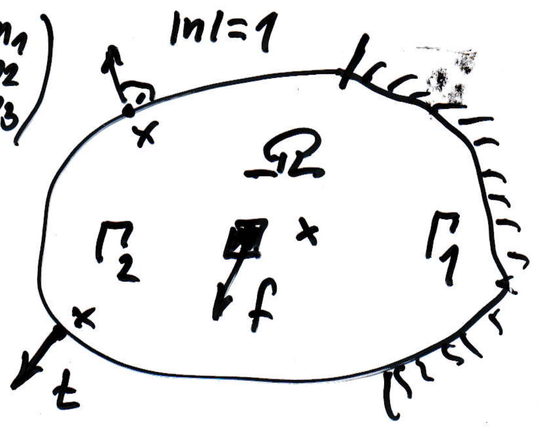
2nd Kind: $\sum_{j=1}^{3} \sigma_{ij}(u(x)) n_j(x) = t_i(x) \quad \forall x \in \Gamma_2 = \Gamma_N = \Gamma_t$

where

$X := \{ v = (v_1, v_2, v_3)^T :$
$v_i \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2) \cap$
$\cap\, C(\Omega \cup \Gamma_1), \; i = \overline{1,3} \}$

input data $\{D_{ijkl}, f, t, \bar{u}, \Omega, \Gamma, \Gamma_1, \Gamma_2\}$ sufficiently smooth!

$n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$, $|n| = 1$



- **Exercises:** for Sect. 1.2.2

$\boxed{\text{Ex 1.7}}$ Show that (8) is equivalent to LAMÉ's system of PDEs: (isotrop)

$$-\mu \Delta u(x) - (\lambda+\mu) \nabla \operatorname{div} u(x) = f(x), \quad x \in \Omega,$$

+ BC: $u = \bar{u}$ on Γ_1 and $\sigma \cdot n = t$ on Γ_2,

with given $f = (f_1, f_2, f_3)^T$, $t = (t_1, t_2, t_3)^T$ and

$$\Delta = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \text{ - vector Laplace,}$$

∇ = grad = gradient, div = divergence

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after T04d

$\boxed{\text{Ex 1.8}}$ = Exercise 07 of Tutorial 2

Show that, for the first linear elasticity BVP, the following properties hold: $(\Gamma = \Gamma_1, \Omega \subset \mathbb{R}^3)$

1) $a(\cdot,\cdot)$ is symmetric, i.e. $a(u,v) = a(v,u)$ $\forall u,v \in \bar{V}$,
2) $a(\cdot,\cdot)$ is non-negative, i.e. $a(v,v) \geq 0$ $\forall v \in \bar{V}$,
3) $a(\cdot,\cdot)$ is positive on $\bar{V}_0 = \{v \in \bar{V} = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_1\}$,
   i.e. $a(v,v) > 0$ $\forall v \in \bar{V}_0 : v \not\equiv 0$.

Properties 1) and 2) yield the equivalence of the VF $(9)_{VF}$ and the MP $(9)_{MP}$.

$\boxed{\text{Ex 1.9}}$ = Exercise 08 of Tutorial 2

Show that, for the first linear elasticity BVP $(\Gamma_1 = \Gamma)$, in the case of isotropic and homogeneous materials, the assumptions of Lax-Milgram's theorem are fulfilled! Compute μ_1 and μ_2!

Hints: for proving the $\bar{V}_0$-ellipticity!

✓ 1) $a(v,v) \geq 2\mu \int_\Omega \sum_{i,j=1} (\varepsilon_{ij}(v))^2 dx$

2) KORN's inequality: $\int_\Omega \sum_{i,j} (\varepsilon_{ij}(v))^2 dx \geq c_K |v|^2_{H^1(\Omega)}$ $\forall v \in \bar{V}_0$?

✓ 3) Friedrich's inequality.

Derivation of the VF in analogy to Section 1.2.1:

① $V_0 := \{ v = (v_1, v_2, v_3)^T \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_1 \}$

② $\int_\Omega [- \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial \sigma_{ij}}{\partial x_j} v_i] dx = \int_\Omega \sum_{i=1}^3 f_i v_i \, dx \quad \forall v \in V_0$

③ $\underbrace{\int_\Omega \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} \frac{\partial v_i}{\partial x_j} dx}_{= \frac{1}{2}(\int_\Omega + \int_\Omega) = \int_\Omega \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij}(u) \varepsilon_{ij}(v) dx} - \int_\Gamma \sum_{i=1}^3 \underbrace{\sum_{j=1}^3 \sigma_{ij} n_j}_{=: \sigma_{n,i}} v_i \, ds = \int_\Omega f^T v \, dx$

$\sigma_{n,i}$ = i-th component of the normal stress vector

$\sigma_{ij} = \sigma_{ji}$

④ $\int_\Gamma \sum_{i=1}^3 \underbrace{\left[\sum_{j=1}^3 \sigma_{ij} n_j \right]}_{\text{natural}} \underbrace{v_i}_{\text{essential}} ds =$

you can impose: natural essential boundary conditions

$= \int_{\Gamma_1} \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} n_j \underbrace{v_i}_{0} ds + \int_{\Gamma_2} \sum_{i=1}^3 t_i v_i \, ds$

⑤ $V_g := \{ v \in V : v = \bar{u} \text{ on } \Gamma_1 \} = V_0$

V_g = Lin. manifold of admissible displacements

wlg: $\bar{u} = 0$ (otherwise: homogenization)

(1)$_g$ $\boxed{\text{Find } u \in V_g : \ a(u,v) = \langle F, v \rangle \ \forall v \in V_0}$

→ T04d

→ T04d

RESULT: Variational Formulation (VF)

$(9)_{VF}$ = VF of (8)

Find $u \in V_g = V_0 := \{ v \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_1 \}$

$$a(u, v) = \langle F, v \rangle \quad \forall v \in V_0$$

where

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{3} \sigma_{ij}(u(x))\, \varepsilon_{ij}(v(x))\, dx = \int_\Omega \sigma^T(u)\, \varepsilon(v)\, dx$$

$$= \int_\Omega \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} D_{ijkl}\, \varepsilon_{kl}(u)\, \varepsilon_{ij}(v)\, dx = \int_\Omega \sigma^T(u)\, D\, \varepsilon(v)\, dx$$

D: tensor of elastic coefficients

$$\boxed{\text{isotrop: } D_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}$$

$$= \int_\Omega \Big\{ \lambda \underbrace{\sum_{k=1}^{3} \varepsilon_{kk}(u)}_{= \operatorname{div} u} \underbrace{\sum_{i=1}^{3} \varepsilon_{ii}(v)}_{= \operatorname{div} v} + 2\mu \sum_{i,j=1}^{3} \varepsilon_{ij}(u)\, \varepsilon_{ij}(v) \Big\} dx$$

$$\langle F, v \rangle = \int_\Omega \sum_{i=1}^{3} f_i v_i\, dx + \int_{\Gamma_2} \sum_{i=1}^{3} t_i v_i\, ds = \int_\Omega f^T v\, dx + \int_{\Gamma_2} t^T v\, ds,$$

with given $f = (f_1, f_2, f_3)^T \in [L_2(\Omega)]^3$, $t = (t_1, t_2, t_3)^T \in [L_2(\Gamma_2)]^3$.

Minimization problem (MP):

$(9)_{MP}$

Find $u \in V_g : J(u) = \inf_{v \in V_g} J(v)$

with the RITZ energy functional

$$J(v) = \underbrace{\frac{1}{2} \int_\Omega \sum_{i,j=1}^{3} \sigma_{ij}(v)\, \varepsilon_{ij}(v)\, dx}_{\substack{= \text{ deformation energy} \\ \text{(inner energy)}}} - \underbrace{\left(\int_\Omega f^T v\, dx + \int_{\Gamma_2} t^T v\, ds \right)}_{\substack{= \text{ potential energy} \\ \text{of the ext. forces}}}$$

Remark 1.5:

1. We refer to the Literature (e.g. [Ciarlet] pp. 23-28 or Lecture Notes on "Solid Mech.") for the discussion of existence and uniqueness of the solution of the mixed BVP (meas $\Gamma_1 > 0$ and meas $\Gamma_2 > 0$) and of the 2nd BVP ($\Gamma_2 = \Gamma$).
2. The basic tool for proving $\exists$ and $!$ is again the Lax-Milgram-Theorem.
   In order to prove V_0-ellipticity, we need
   - $D(\cdot)$ is uniformly SPD:
     $$\lambda_{min}(D)\,\|\varepsilon(v)\|^2_{L_2(\Omega)} \le a(v,v) \le \lambda_{max}(D)\,\|\varepsilon(v)\|^2_{L_2(\Omega)} \quad \forall v \in V$$
     with
     $$\lambda_{\substack{min\\max}}(D) = \substack{min\\max}_{x\in\bar\Omega}\ \underbrace{\lambda_{\substack{min\\max}}(x)}_{\text{EV of } D(x)},$$
   - KORN's inequality:
     $$(10)\qquad \|v\|_{H^1(\Omega)} \le c_K \Big[\sum_{i,j=1}^{3} \|\varepsilon_{ij}(v)\|^2_{L_2(\Omega)} + \sum_{i=1}^{3}\|v_i\|^2_{L_2(\Omega)}\Big]^{1/2}$$
     $$\forall v \in V := [H^1(\Omega)]^3,$$
   - FRIEDRICHS' inequality (mixed BVP),
   - the Knowledge of the rigid body motion subspace (2nd BVP)
     $$\operatorname{Ker} a(\cdot,\cdot) = \operatorname{Ker} A_{Lamé}$$
     $$:= \{ a \times x + b : a, b \in \mathbb{R}^3 \}$$
     $$= \operatorname{span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-x_2\\x_1\\0\end{pmatrix}, \begin{pmatrix}0\\-x_3\\x_2\end{pmatrix}, \begin{pmatrix}-x_3\\0\\x_1\end{pmatrix} \right\}$$
3. It holds (mms):
   $$(11)\qquad \varepsilon(v) = \mathbb{0} \iff v \in \operatorname{Ker} A_{Lamé}$$

1.2.3. BVP for Scalar Elliptic PDEs of 4th Order: The First Biharmonic BVP

▣ The Classical Formulation of the 1st biharmonic BVP:

(12) Find $u \in X := C^4(\Omega) \cap C^1(\overline{\Omega})$:

$$\Delta^2 u(x) := \frac{\partial^4 u}{\partial x_1^4}(x) + 2\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2}(x) + \frac{\partial^4 u}{\partial x_2^4}(x) = f(x) \quad \forall x = (x_1, x_2) \in \Omega,$$

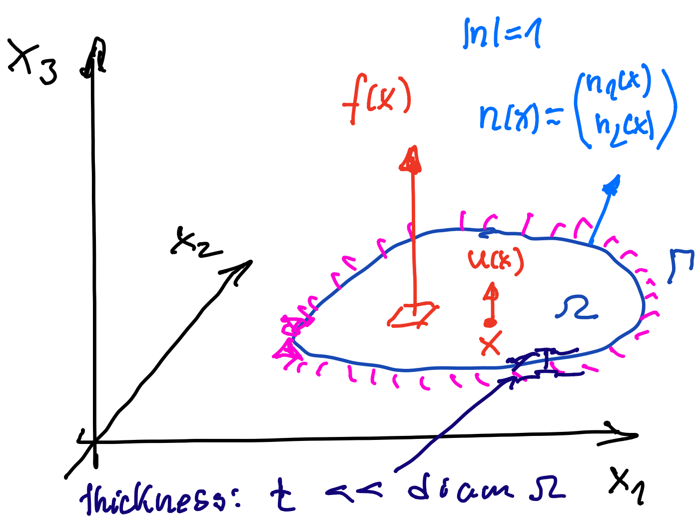
+ BC': $u(x) = 0 \quad \forall x \in \Gamma = \partial\Omega,$

$\frac{\partial u}{\partial n}(x) = 0 \quad \forall x \in \Gamma = \partial\Omega,$

with given $f \in C(\Omega)$ and $\Omega \subset \mathbb{R}^2$ *,

$\Gamma = \partial\Omega$ - sufficiently smooth.

(12) is a model for describing the vertical bending u of clamped, homogeneous and isotropic plates under vertical loading f.



BVP (12) can be derived from 3d linear elasticity problem (8) by means of the so-called KIRCHHOFF hypotheses, see also Braess [AL1].

Derivation of the Variational Formulation: ① - ⑤

① $\mathring{V}_0 = \{ v \in V = H^2(\Omega) : v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma \} = \mathring{H}^2(\Omega) = \mathring{W}_2^1(\Omega)$,

② $\int_\Omega \Delta^2 u \cdot v \, dx = \int_\Omega f \cdot v \, dx \quad \forall v \in \mathring{V}_0$

2x partial integration

③ $\int_\Omega \Delta(\Delta u) \cdot v \, dx = -\int_\Omega \nabla \Delta u \cdot \nabla v \, dx + \int_\Gamma \frac{\partial \Delta u}{\partial n} v \, ds =$

$= \int_\Omega \Delta u \cdot \Delta v \, dx - \int_\Gamma \underbrace{\Delta u}_{n.} \underbrace{\frac{\partial v}{\partial n}}_{e.} ds + \int_\Gamma \underbrace{\frac{\partial \Delta u}{\partial n}}_{n.} \underbrace{v}_{e.} ds$ BCs

④ No natural BC ($\Delta u = g_2$ and $\partial_n \Delta u = g_3$)! $\quad \partial_n = \frac{\partial u}{\partial n}$

⑤ $V_g = \{ v \in V : v = g_0 := 0 \text{ and } \partial_n v = g_1 := 0 \text{ on } \Gamma_1 \} = V_0 = \mathring{H}^2(\Omega)$

Result: Variational Formulation

(13) $\boxed{\begin{array}{l} \text{Find } u \in \mathring{V}_g = \mathring{V}_0 = \mathring{H}^2(\Omega) : a(u,v) = \langle F, v \rangle \ \forall v \in \mathring{V}_0, \\ \text{where } a(u,v) = \int_\Omega \Delta u \cdot \Delta v \, dx \text{ and } \langle F, v \rangle = \int_\Omega f v \, dx, \\ \quad f \in L_2(\Omega) \text{ given}, \ \Gamma = \partial\Omega \in C^{0,1}, \ \Omega \neq \text{ and Lip.} \end{array}}$

$\boxed{\text{E 1.10}}$ see T05, p. L04-08.

Remark 1.6:

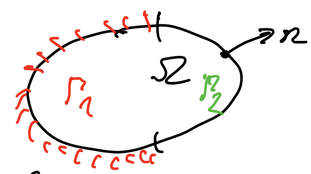
1. Step ③ of the derivation of the VF yields the following possible BC:
   - essential : $u = g_0$ on Γ
   - essential : $\partial_n u = g_2$ on Γ
   - natural : $\Delta u = g_3$ on Γ
   - natural : $\partial_n \Delta u = g_4$ on Γ

   possible combinations

2. Possible BVP for the biharmonic PDE $\Delta^2 u = f$:

   1st BVP: $u = g_0$ and $\partial_n u = g_1$ on Γ (pure Dirichlet BVP)
   2nd BVP: $u = g_0$ and $\Delta u = g_2$ on Γ
   3rd BVP: $\partial_n u = g_1$ and $\partial_n \Delta u = g_3$ on Γ
   4th BVP: $\Delta u = g_2$ and $\partial_n \Delta u = g_3$ on Γ (pure Neumann)

   mixed BVP e.g. $u = \partial_n u = 0$ on Γ_1
   $\Delta u = \partial_n \Delta u = 0$ on Γ_2



3. The KIRCHHOFF plate bilinear form

$$a(u,v) := \int_\Omega \Big\{ \Delta u \Delta v + (1-\sigma)\Big[2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} \Big] \Big\} dx$$

   coincides with the biharmonic bilinear form (13)

$$a(u,v) = \int_\Omega \Delta u \cdot \Delta v \, dx$$

   ONLY in the case of the 1st. biharmonic BVP (mans). Therefore, there are other natural BC for the plate bending problem (mms). The material parameter $\sigma \in (0,1)$ is called Poisson's coefficient.

$\boxed{\textbf{E 1.11}}$ Derive the VF of the BVPs given in Remark 1.6.2!
See also T05 on p. L04-09 (next page)!

Exercises: for Section 1.2.3

$\boxed{\text{Ex 1.10}}$ = Exercise 11 in Tutorial 02

Show that the following properties hold for the first biharmonic BVP

(13) Find $\overset{\circ}{H}{}^2(\Omega)$: $\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v dx \;\; \forall v \in \overline{V}_0 = \overset{\circ}{H}{}^2(\Omega)$:

1) $F \in V_0^* = H^{-2}(\Omega) = W_2^{-2}(\Omega)$,

2) $a(\cdot,\cdot): \overline{V}_0 \times \overline{V}_0 \to R^1$ is a bilinear form:

2a) $\exists \mu_1 = const > 0 : a(v,v) \geq \mu_1 \|v\|_2^2 \;\; \forall v \in \overline{V}_0 = \overset{\circ}{H}{}^2(\Omega)$,

2b) $\exists \mu_2 = const > 0 : |a(u,v)| \leq \mu_2 \|u\|_2 \|v\|_2 \;\; \forall u, v \in \overline{V}_0$,

2c) $a(u,v) = a(v,u) \;\; \forall u,v \in \overline{V}_0$.

These properties yield:

- $\exists!$ (Lax-Milgram) (Ass. 1) - 2b), but not 2c)
- Equivalenze of $(13)_{VF}$ to the MP $(13)_{MP}$!

where $\|\cdot\|_2 := \|\cdot\|_{H^2(\Omega)}$.

$\boxed{\text{Ex 1.11}}$ = Exercise

Derive the variational formulations of the BVP given in Remark 1.6.2! Investigate existence and uniqueness of generalized solutions (LGM)! Without loss of generality (homogenization), you can assume that the essential BC are homogeneous.