- **Remark 1.2:**
  1. The solution $u \in \tilde{V}_g$ of (6) is called weak or generalized solution.
  2. The assumptions imposed on the data of (6) can be weakened (! integrals must exist !), e.g.

(7)
$$
\begin{cases}
1)\ a_{ij}, b_i, c \in L_\infty(\Omega),\ \alpha \in L_\infty(\Gamma_3), \\
2)\ f \in L_2(\Omega),\ g_i \in L_2(\Gamma_i),\ i = 2,3, \\
3)\ g_1 \in H^{1/2}(\Gamma_1) := \gamma_{0,\Gamma_1} H^1(\Omega),\ \text{i.e. } \exists \tilde{g}_1 \in H^1(\Omega): \\
\quad \tilde{g}_1|_{\Gamma_1} := \gamma_{0,\Gamma_1} \tilde{g}_1 = g_1 \\
4)\ \Omega \subset \mathbb{R}^d\ {}^*: \Gamma = \partial\Omega \in C^{0,1} \text{ (Lipschitz cont.)}, \\
5)\ \text{uniform ellipticity}: \exists \bar{\mu}_1 = \text{const} > 0: \\
\quad \left.\begin{array}{l} \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \bar{\mu}_1 |\xi|^2 \ \forall \xi \in \mathbb{R}^d \\ a_{ij}(x) = a_{ji}(x)\ \forall i,j = \overline{1,d} \end{array}\right\} \dot{\forall} x \in \Omega \text{ a.e.}
\end{cases}
$$

↓ homg. ansatz: $u = \tilde{g}_1 + w$

* = bounded

$\dot{\forall}$ = for almost all

Further weakening of the assumptions: $d = 1,2,3$

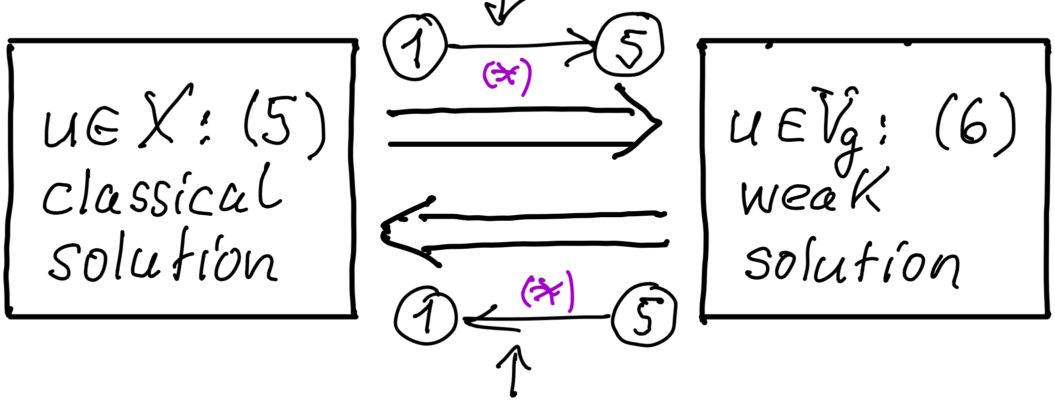
e.g. $\left.\begin{array}{l} b_i \in H^1(\Omega) \Longrightarrow b_i \in L_4(\Omega) \\ v \in H^1(\Omega) \Longrightarrow v \in L_4(\Omega) \end{array}\right\} \Rightarrow b_i v \in L_2(\Omega).$

↳ Ch 2: Embedding theorems

L03 - 02

3. Relations between classical and weak solutions:

$u \in X \cap V_g \cap H^2(\Omega)$



Ass.:

- $u \in V_g \cap X \cap H^2(\Omega)$
- classical assumptions imposed on the data in $\overline{\Omega}$

(*) Existence of the integrals and feasibility of the partial integration must be ensured!

$$u \in X := C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1)$$

$$\not\Downarrow$$

$u \in H^1(\Omega)$! (mms)

L03-03

Exercises:

Ex 1.1. = Exercises 01-02 of Tutorial 1

Formulate the classical assumptions which we have to impose on the data $\{a_{ij}, b_i, c, \alpha, f, g_i, \Omega\}$ of (5)! Provide sufficient conditions in order to ensure that a weak solution $u \in V_g \cap X \cap H^2(\Omega)$ of (6) is also a solution of (5) in the classical sense!
Consider first the Dirichlet problem for the Poisson equation as training:

$(5)_{\text{Poisson}}$

> Find $u \in X := C^2(\Omega) \cap C^1(\overline{\Omega})$:
> $-\Delta u = f$ in Ω, $u = g$ on $\Gamma = \partial\Omega \in C^2$

? $\Downarrow$ $\Uparrow$?

> Find $u \in V_g := \{v \in V = H^1(\Omega) : v = g \text{ on } \Gamma\}$:
> $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in V_0 = \overset{\circ}{H}{}^1(\Omega)$

Ex 1.2 = Exercise 03 of T1:

Show that in cases a)-c) the Acs. of L&M-Th. are satisfied and compute μ_1 and μ_2! We assume (7) and

a) $b_i = 0$, $c(x) \geq 0 \; \forall x \in \Omega$, $\alpha(x) \geq 0 \; \forall x \in \Gamma_3$, $meas_{d-1}(\Gamma_1) > 0$;

b) $b_i = 0$, $c = 0$, $\alpha(x) \geq \underline{\alpha} = const > 0 \; \forall x \in \Gamma_3$, $meas_{d-1}(\Gamma_3) > 0$, $\Gamma_1 = \emptyset$;

c) $b_i = 0$, $c(x) \geq \underline{c} = const > 0 \; \forall x \in \Omega$, $\Gamma = \Gamma_2$ (i.e. $\Gamma_1 = \Gamma_3 = \emptyset$)!

L03 - 04

$\boxed{\text{Ex. 1.3.}}$ = Exercise 04 of T1:

In addition to the assumption (7), let us assume that $c(x) \geq \underline{c} = \text{const} > 0 \ \forall x \in \Omega$, $\Gamma_1 = \Gamma_3 = \emptyset$, $b_i \not\equiv 0$.
Provide sufficient conditions for the coefficients b_i such that the ass. of the LeM-Th. are satisfied!

<u>Hint:</u> For estimating the convection term
$\sum_{i=1}^{d} \int_\Omega b_i \frac{\partial u}{\partial x_i} v \, dx$, make use of ε-inequality (Young's)
$|ab| \leq \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2 \quad \forall a, b \in \mathbb{R} \quad \forall \varepsilon > 0.$

$\boxed{\text{Ex. 1.4.}}$ = Exercise 05 of T1:

Derive the variational formulation of the pure Neumann problem for the Poisson equation

(*) $-\Delta u = f$ in Ω, $\frac{\partial u}{\partial n} := \nabla u \cdot n = 0$ on $\Gamma = \partial\Omega$,

and discuss the question of the existence and the uniqueness of the weak solution of (*)!

<u>Hints:</u> Obviously, $u(x) + c$ with an arbitrary constant $c \in \mathbb{R}^1$ solves (*) provided that u is a solution of the BVP (*)!
There are the following ways to analyze the solvability of (*):

1) Set up the VF in $\tilde{V} = H^1(\Omega)$ and apply Fredholm's theory!
2) Set up the VF in $V = H^1(\Omega)|_{Ker}$ with $Ker = \{c : c \in \mathbb{R}^1\} = \mathbb{R}^1$ and apply the Lax-Milgram Theorem!
3) Consider the VF: Find $u \in \tilde{V} = H^1(\Omega)$:
$\int_\Omega \nabla u \cdot \nabla v \, dx + \beta \int_\Omega u \, dx \int_\Omega v \, dx = \int_\Omega f v \, dx \ \forall v \in \tilde{V}$, $\beta \in \mathbb{R}^1_+$!

arbitrary, but fixed

L03 - 05

$\boxed{\text{Ex. 1.5}}$ = Exercise 06* of T1:

Derive the VF of the Dirichlet problem for the Helmholtz equation

(**) $-\Delta u - \omega^2 u = f$ in Ω, $u = 0$ on Γ !

Then discuss the existence and uniqueness of a weak solution of the BVP (**), where ω^2 is a given positive constant !

$\boxed{\text{Ex. 1.6.}}$ Let us consider the Dirichlet BVP for determining the 3rd z-component $u(x_1,x_2) = A_z(x,y) = A_3(x_1,x_2)$ of the magnetic vector potential for a plane magnetic field problem (e.g. electrical machine):

$$(1) \quad \begin{cases} -\operatorname{div}\left(\frac{1}{\mu}\nabla u\right) = J_3 - \frac{\partial H_2}{\partial x_1} + \frac{\partial H_1}{\partial x_2} & \text{in } \Omega \subset \mathbb{R}^2 \ *, \\ u = 0 & \text{on } \Gamma = \partial\Omega . \end{cases}$$

Derive the VF: Find $u \in V_g : a(u,v) = \langle F, v\rangle \ \forall v \in V_0$ of (1), and show that there exists a unique weak solution $u \in V_g = V_0$ provided that the following <u>assumptions</u> are fulfilled:

1) $\mu \in L_\infty(\Omega) : 0 < \underline{\mu} \le \mu(x) \le \overline{\mu} \ \forall^a x \in \Omega$, with positive constants $\underline{\mu}$ and $\overline{\mu}$,
2) $J_3 \in L_2(\Omega)$,
3) $H_1, H_2 \in L_2(\Omega)$,
4) $\Omega \subset \mathbb{R}^2 \ *$ and $\partial\Omega \in C^{0,1}$.

- **Sobolev's norm equivalence theorem** is a powerful tool for proving V_0-ellipticity and V_0-Υ of $a(\cdot,\cdot)$:

Theorem 1.3:

Ass.: Let $\Omega \subset \mathbb{R}^d$ be a bounded Lip-domain, i.e. $\partial\Omega \in C^{0,1}$, and let $1 \le p < \infty$, $k \in \{1, 2, \dots\}$, and let

$f_i : W_p^k(\Omega) \longmapsto \mathbb{R}^1$ ($\mapsto [0,\infty)$), $i = 1, 2, \dots, \ell$

be a system of semi-norms such that

a) $\exists\, c_i = \text{const} > 0 : 0 \le f_i(u) \le c_i \|u\|_{W_p^k(\Omega)} \quad \forall u \in W_p^k(\Omega)$,

b) $\left.\begin{array}{l} f_i(v) = 0 \ \forall i = \overline{1,\ell}, \\ v \in \mathcal{P}_{k-1} := \{ \sum\limits_{|\alpha| \le k-1} c_\alpha x^\alpha \} \end{array}\right\} \Longrightarrow v = 0.$

St.: Then $\exists\, \underline{c}, \overline{c} = \text{const} > 0$:

$$\underline{c}\, \|u\|^*_{W_p^k(\Omega)} \le \|u\|_{W_p^k(\Omega)} \le \overline{c}\, \|u\|^*_{W_p^k(\Omega)} \quad \forall u \in W_p^k(\Omega),$$

where
$$\|u\|^*_{W_p^k(\Omega)} := \Big(\sum_{i=1}^{\ell} f_i^p(u) + |u|^p_{W_p^k(\Omega)} \Big)^{1/p},$$

$$|u|_{W_p^k(\Omega)} := \Big(\sum_{|\alpha| = k} \int_\Omega |\partial^\alpha u|^p \, dx \Big)^{1/p},$$

$$\|u\|_{W_p^k(\Omega)} := \Big(\underbrace{\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^p \, dx}_{\text{standard } W_p^k\text{-norm}} \Big)^{1/p}.$$

Proof: see Proof of Th. 2.13 of these lectures, OR LN Numerik I, Satz 3.1, p. 69 ff. $\square$

Corollary 1.4:

If $\text{meas}_{d-1}(\Gamma_3) > 0$, then $\exists\, c = \text{const} > 0$:

$$\int_\Omega |\nabla u|^2 dx + \int_{\Gamma_3} u^2 ds \geq c \|u\|^2_{W_2^1(\Omega)} \quad \forall u \in W_2^1(\Omega).$$

$H^1(\Omega)$

Proof: directly follows from Theorem 1.3:

Indeed: $p = 2,\ K = 1,\ \ell = 1$:

$$f_1(u) := \sqrt{\int_{\Gamma_3} u^2 ds}$$

$\Longrightarrow \quad f_1 : W_2^1(\Omega) \to \mathbb{R}^1$ is obviously a semi-norm:

a) $0 \leq f_1(u) = \|u\|_{L_2(\Gamma_3)} \leq \|u\|_{L_2(\Gamma)}$

$\leq c_T \|u\|_{W_2^1(\Omega)}$,

due to trace theorem, see Ch. 2.

b) $$\left.\begin{array}{l} f_1(v) = \int_{\Gamma_3} v^2 ds = 0 = v^2|\Gamma_3| \\ v \in P_0 = \mathbb{R},\ v = \text{const} \end{array}\right\} \Rightarrow v = 0.$$

Th. 1.3 delivers that $\exists\, \underline{c}, \bar{c} = \text{const} > 0$

$$\underline{c}^2 \left[\int_{\Gamma_3} u^2 ds + \int_\Omega |\nabla u|^2 dx\right] \leq \|u\|^2_{W_2^1(\Omega)} \leq \bar{c}^2 \left[\qquad\right],$$

i.e. $c^2 = \bar{c}^{-2}$. $\square$