1.2. Examples: BVP for Elliptic PDEs

1.2.1. BVP for Scalar 2nd-Order Elliptic PDEs

- **Repition:** → "PDEs": Chapter 10

[WZ1: Ch.2$^{d=1}$, Ch.6] → "Nu PDEs": Sect. 1.1–1.2 (1d), 1.5.

- **Classical Formulation (in divergence form):**

(5) Find $u \in X := C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1)$:

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x) = f(x),\ x\in\Omega$$

$$\overset{\nabla\cdot}{} \qquad \overset{\nabla}{} \qquad \overset{\nabla}{}$$

$$-\operatorname{div}(A \operatorname{grad} u) + b\cdot\operatorname{grad} u + cu = f \text{ in } \Omega$$

$$-\nabla\cdot(A\nabla u) + b\cdot\nabla u + cu = f \text{ in } \Omega$$

opera. ↑ A double notation ↓ coeff. matrix

A ↑ unif. SPD

+ BC: 1st Kind (Dirichlet): $u(x) = g_1(x),\ x\in\Gamma_1 = \Gamma_D$,

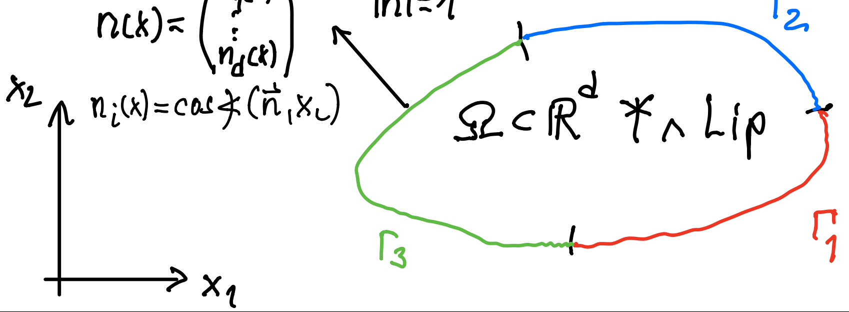
2nd Kind (Neumann): $\frac{\partial u}{\partial N}(x) = g_2(x),\ x\in\Gamma_2 = \Gamma_N$,

3rd Kind (Robin): $\frac{\partial u}{\partial N}(x) + \alpha(x)u(x) = g_3(x),\ x\in\Gamma_3 = \Gamma_R$

flux: $-\frac{\partial u}{\partial N} = \alpha(x)(u(x) - u_A(x))$ $\qquad g_3(x) = \alpha(x)u_A(x)$

where $\frac{\partial u}{\partial N}(x) := A\nabla u\cdot n$ – co-normal derivative

$n(x) = \begin{pmatrix} n_1(x) \\ \vdots \\ n_d(x)\end{pmatrix}$ $\qquad |n| = 1$



Remerk 1.1:

1. $u \in X$: (5) is called **classical solution** of the BVP (5)!
2. Note the classical assumptions imposed on the data $\{a_{ij}, b_i, c, \alpha, f, g_i, \Omega\} \to$ suff. smooth! (mms)
3. **Uniform ellipticity** in Ω:
   - $a_{ij}(x) = a_{ji}(x), \forall x \in \Omega \cup \Gamma_2 \cup \Gamma_3 \; \forall i,j = \overline{1,d} \quad (A = A^T)$,
   - $\exists \bar{\mu}_1 = const > 0 : \forall \xi \in \mathbb{R}^d \; \forall x \in \Omega \cup \Gamma_2 \cup \Gamma_3$:

$$\bar{\mu}_2 |\xi|^2 \geq (A(x)\xi, \xi) = \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq \bar{\mu}_1 |\xi|^2$$

BVP (5) is a mathematical model for (e.g.)

1. Stationary heat conduction / transport problems,
2. Stationary diffusion-convection-reaction problems,
3. Potential problems:

a) electrical scalar problems: $E = \nabla u$

Find $u \in X := C^2(\Omega) \cap C(\bar{\Omega})$: $-\Delta u = f$ in Ω, $u = g_1$ on $\Gamma = \Gamma_1$,

b) magnetic (vector) potential: $B = \operatorname{curl} A$ (2D: $d = 2$)

Find $u = A_3 \in X := C^2(\Omega) \cap C^1(\bar{\Omega})$:

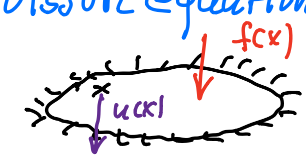
$$-\operatorname{div}(\nu(x) \nabla u(x)) = J_3(x) - \frac{\partial H_2}{\partial x_1}(x) + \frac{\partial H_1}{\partial x_2}(x), \; x \in \Omega \subset \mathbb{R}^2,$$

$+$ BC: $u(x) = 0, \; x \in \Gamma_1 = \Gamma = \partial\Omega$,

c) membrane problem (= Poisson equation):

Find $u \in X$: $-\Delta u = f$ in Ω

$u = 0$ on $\partial\Omega = \Gamma_1$



clamped membrane

$f(x,t) = f(x) e^{i\omega t}$

$\frac{\partial^2 u}{\partial t^2} - \Delta u = f$

$u(x,t) = u(x) e^{i\omega t}$

d) Helmholtz-equation:

Find $u \in X$: $-\Delta u - k^2 u = f$ in Ω, $u = 0$ on $\Gamma_1 = \Gamma$,

with $k^2 = \omega^2 / a^2$ ($\Leftarrow$ harmonic excitation of vibrations)

■ Variational (= weak = generalized) Formulation:

- Formal Procedure for Deriving of VF (1) of (5):

① Choose the space of test functions:

$$V_0 = \{ v \in V = H^1(\Omega) = W_2^1(\Omega) : \; v = \gamma_0 v\big|_{\Gamma_1} = 0 \text{ on } \Gamma_1 \},$$

↳ basic space for lin., scalar, 2nd-order ell. PDEs

② Multiply the PDE (5) by a test function $v \in V_0$, and integrate over the computational domain Ω:

$$\int_\Omega \Big(-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu \Big) v\, dx = \int_\Omega f v\, dx \quad \forall v \in V_0,$$

③ Integrate by parts in the main part ($(a_{ij} u_{,j})_{,i}$):

$$\boxed{\int_\Omega \frac{\partial w}{\partial x_i} v\, dx = -\int_\Omega w \frac{\partial v}{\partial x_i} dx + \int_\Gamma w v n_i\, ds}$$

$$\int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} v + cuv \Big) dx - \int_\Gamma \underbrace{\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j} n_i}_{=:\frac{\partial u}{\partial N}} v\, ds =$$

$$= \int_\Omega f v\, dx \quad \forall v \in V_0$$

nat. ess.

④ Incorporate the natural BC (Γ_2, Γ_3) into the VF:

$$\int_\Gamma \frac{\partial u}{\partial N} v\, ds = \int_{\Gamma_1} \frac{\partial u}{\partial N} \underbrace{v}_{=0} ds + \int_{\Gamma_2} g_2 v\, ds + \int_{\Gamma_3} (g_3 - \alpha u) v\, ds,$$

⑤ Define the linear manifold of admissible functions where we search for the solution u:

$$V_g := \{ v \in V = H^1(\Omega) : \; v = g_1 \text{ on } \Gamma_1 \}.$$

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- <u>Result:</u> Variational Formulation (VF)

(6) Find $u \in V_g$: $a(u,v) = \langle F, v \rangle \;\; \forall v \in V_0$,

with

$$a(u,v) := \int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} v + c u v \Big) dx + \int_{\Gamma_3} \alpha u v \, ds,$$

$$A \nabla u \cdot \nabla v + b \cdot \nabla u \, v + c u v$$

$$\langle F, v \rangle := \int_\Omega f v \, dx + \int_{\Gamma_2} g_2 v \, ds + \int_{\Gamma_3} g_3 v \, ds,$$

$$V_g := \{ v \in V = H^1(\Omega) : v = g_1 \text{ on } \Gamma_1 \}, \qquad \gamma_{0\Gamma_1} v = g_1$$

$$V_0 := \{ v \in V = H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \}. \qquad \gamma_{0\Gamma_1} v = 0$$