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NuEPDE

NUMERICAL METHODS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Numerik elliptischer Probleme

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1. Variational Formulation of Multidimensional Elliptic Boundary Value Problems and their Analysis

1.1. Abstract Theory

- **Repitition:**
  - → Lectures "PDEs": Chapter 10
  - → Lectures "NuPDEs": Sect. 1.1-1.2, [WZ: Ch. 3]

- Let us consider the abstract variational problem:

$(1)_g$ Find $u \in V_g$: $a(u,v) = \langle F, v \rangle \;\; \forall v \in V_0$

homogenization $\updownarrow$ $\qquad u = \underset{\text{given}}{g} + \underset{\text{unknown}}{w} \in V_g = g + V_0 \subset V$

$(1)_0$ Find $w \in V_0$: $a(w,v) = \langle \hat{F}, v \rangle := \langle F, v \rangle - a(g,v)$

wlg $\qquad u \in V_0$: $a(u,v) = \langle F, v \rangle \;\; \forall v \in V_0$,

where

$a(\cdot,\cdot): V \times V \longrightarrow \mathbb{R}$ ($\mathbb{C}$) – continuous bilinear form (sesquilinear form) (given),

$F(\cdot) = \ell(\cdot) := \langle F, \cdot \rangle : V_0 \to \mathbb{R}$ – cont. linear form (given),

$\langle \cdot, \cdot \rangle : V_0^* \times V_0 \to \mathbb{R}$ – duality product,

V – Hilbert-space (basic space): $\|\cdot\| = \|\cdot\|_V$, $(\cdot,\cdot) = (\cdot,\cdot)_V$

$V_0 \subset V$ – closed, non-trivial subspace (= test space)

$V_g = g + V_0 := \{ v \in V : v = g + w \text{ with } w \in V_0 \text{ and given } g \in V \}$

– linear manifold (hyperplane) of admissible functions.

The main result: Lax-Milgram's theorem (Lemma):

→ see Th. 10.1. in "PDEs" or Th. 1.28 in "NuPDEs" [WZ1: p. 25]

Ass.: 0. $V_0 \subset V$ with $\|\cdot\| = (\cdot,\cdot)^{1/2}$, see above.

1. $F \in V_0^*$,
2. $a(\cdot,\cdot) : V_0 \times V_0 \to$ bilinear form:

2a) V_0-elliptic: $\exists\, \mu_1 = const > 0 : \mu_1 \|v\|^2 \le a(v,v) \;\forall v \in V_0$

2b) V_0-bounded (continuous): $\exists\, \mu_2 = const > 0$:
$$|a(u,v)| \le \mu_2 \|u\| \, \|v\| \quad \forall u,v \in V_0.$$

St.: $\exists!\, u \in V_0 : a(u,v) = \langle F, v \rangle \;\forall v \in V_0 \quad (1)_0$

$$(2) \quad \frac{1}{\mu_2} \|\underset{Au}{F}\|_{V_0^*} \le \|\overset{A^{-1}F}{u}\|_{V_0} \le \frac{1}{\mu_1} \|\underset{Au}{F}\|_{V_0^*}.$$

Proof: $(1)_0 \Longleftrightarrow Au = F$ in V_0^* with $A \in L(V_0, V_0^*)$:
$$\langle Au, v \rangle = a(u,v) \quad \forall u,v \in V_0$$

Banach's fix point theorem: $\varrho \in (0, 2\mu_1/\mu_2^2)$:

$$(3) \quad \overset{\text{Cauchy}}{u_{n+1}} = u_n - \varrho\,(JAu_n - JF) \xrightarrow[n\to\infty]{} \overset{\exists}{u} \text{ in } V_0,$$

where $J : V_0^* \longrightarrow V_0$ – Riesz' isomorphism:

$$\langle F, v \rangle = (JF, v) \quad \forall v \in V_0 \;\forall F \in V_0^*$$

bij.

∎

$$\|u - u_{n+1}\| \le q(\varrho)\, \|u - u_n\|$$

with $q(\varrho) := \sqrt{1 - 2\varrho\mu_1 + \varrho^2\mu_2^2} < 1$

Minimization problem: see L. 1.30 / 1.40 in "NuPDEs"

Ass.: 1. $F \in V^*$,

2. Bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}^1$:

(i) symmetric: $a(u,v) = a(v,u) \quad \forall u,v \in V$,

(ii) ~~non-negative~~ positive: $a(v,v) \overset{>}{\geq} 0 \quad \forall v \in V \setminus \{0\}$ $(\exists!)$

St.: Then the VP(1) is equivalent to the MP (4):

$$(1)_g \quad \boxed{\begin{array}{l}\text{Find } u \in V_g : \\ a(u,v) = \langle F, v\rangle \\ \forall v \in V_0\end{array}} \iff \boxed{\begin{array}{l}\text{Find } u \in V_g : \\ J(u) = \inf\limits_{w \in V_g} J(w)\end{array}} \quad (4)_g$$

with the Ritz energy functional $J(v) = \frac{1}{2} a(v,v) - \langle F, v\rangle$.

Proof: mms or see proof of L. 1.40 = L. 1.30 in "NuPDEs".

Question:

Lax-Milgram's theorem only delivers suff. conditions for $\exists!$ of the solution $u \in V_0$ of the operator equation

$(1)_0$ Find $u \in V_0$: $Au = F$ in V_0^*

$\langle Au, v\rangle := a(u,v) = \langle F, v\rangle$

Can we formulate sufficient and necessary conditions for $A : V_0 \to V_0^*$ being an isomorphism (iso), i.e., bijective ($\leftrightarrow$), A and A^{-1} continuous (*), i.e. $(1)_0$ is well-posed ?

$U = V = V_0$ $(\downarrow)$

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- **Answer:** BNB - Theorem:

Banach - Nečas - Babuška - Asiz - Theorem:
Let U and V be Hilbert spaces ($\|\cdot\|_X, (\cdot,\cdot)_X$).
The linear operator (map) $A: U \to V^*$ is an **isomorphism** (bijective, A and A^{-1} continuous (*))
if and only if ($=$ iff $= \Leftrightarrow$) the corresponding bilinear form $a(\cdot,\cdot) := \langle A\cdot, \cdot \rangle_{V^* \times V} : U \times V \to \mathbb{R}^1$
fulfills the following three conditions:

1. continuity $=$ * $=$ sup-sup-condition, i.e.
$\exists \mu_2 = \text{const} > 0 : |a(u,v)| \le \mu_2 \|u\|_U \|v\|_V \quad \forall u \in U \; \forall v \in V,$
2. inf-sup-condition, i.e. $\exists \mu_1 = \text{const} > 0$:

A inj $\wedge$ range A closed
$$\inf_{u \in U \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{a(u,v)}{\|u\|_U \|v\|_V} \ge \mu_1 > 0,$$

A^* inj.
3. $\overset{\text{fix}}{\forall} v \in V \setminus \{0\} \; \exists u \in U : a(u,v) \neq 0.$

Proof: see Lectures NuCM by Walter Zulehner. □

- **Remark:**

1. In the case $U = V = V_0$, the Ass. 2a) and 2b) of Lax-Milgram's theorem obviously imply Ass. 1. - 3. of the BNB-Theorem! (mms)
2. BNB-Theorem is also valid for Banach space setting: U - Banach space, V - reflex. Banach space:
Find $u \in U$: $Au = F$ in V^*.