BEM-based Finite Element Approaches on Polytopal Meshes-Anisotropic Case

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Overview

Outline

Preliminary Definitions

Let

- $K \subset \mathbb{R}^d, d \in \{2,3\}$ be a polytopal or polyhedral element
- $|K| = meas_d(K) > 0$
- $\mathscr{K}_h$ be the set of polytopal or polyhedral elements in the mesh
- $\mathscr{N}_h$ be the set of all nodes
- $\mathscr{N}_{h,D}$ be the set of all nodes on the Dirichlet boundary $\varGamma_D$
- V_h^k be the approximation space over polytopal mesh of order k, $V_h^1 = V_h$
- ω_z be the neighbourhood of all elements adjacent to the node z
- ω_F be the neighbourhood of all elements adjacent to the face F
- ω_K be the neighbourhood of all elements adjacent to the element K
- $H^s(\Omega)$ be the Sobolev space of order $s \in \mathbb{R}$ over Ω

Preliminary definitions

Definition (Center (Mean) of K)

The center or mean of K is defined as

$$\bar{x}_K = \frac{1}{|K|} \int_K x \, dx.$$

Definition (Covariance matrix of K)

The covariance matrix of K is defined by

$$M_{Cov}(K) = \frac{1}{|K|} \int_K (x - \bar{x}_K)(x - \bar{x}_K)^\top \, dx \in \mathbb{R}^{d \times d}.$$

Preliminary definitions

- $M_{Cov}(K)$ admits an eigenvalue decomposition

$$M_{Cov}(K) = U_K \Lambda_K U_K^\top$$

  with $U_K^\top = U_K^{-1}$ and $\Lambda_K = diag(\lambda_{K,1} \dots \lambda_{K_d})$ where w.l.o.g. $\lambda_{K,1} \geq \dots \geq \lambda_{K,d}$.
- Eigenvectors $u_{K,1}, \dots u_{K,d} \in U_K$ give the characteristic directions of K
- Eigenvalues $\lambda_{K,j}, j = 1, \dots, d$ is the variance in direction of the corresponding eigenvector $u_{K,j}$

Preliminary definitions

Definition (Isotropy, Anisotropy)

An element is called

- isotropic if

$$\frac{\lambda_{K,1}}{\lambda_{K,d}} \approx 1$$

- and anisotropic if

$$\frac{\lambda_{K,1}}{\lambda_{K,d}} \gg 1.$$

Preliminary definitions

Definition (Reference configuration)

For each $x \in K$ we define the mapping

$$x \mapsto \hat{x} = \mathscr{F}(x) = A_K x \tag{1}$$

with $A_K x = \alpha_K \Lambda_K^{-1/2} U_K^\top x$, $\alpha_K > 0$.

$\hat{K} = \mathscr{F}(K)$ is called reference configuration.

Relations between K and $\hat{K}$

Theorem

Under the previously defined transformation, it holds

- $|\hat{K}| = |K||det(A_K)| = \alpha_K^d |K| / \sqrt{\Pi_{j=1}^d \lambda_{K,j}}$,
- $\bar{x}_{\hat{K}} = \mathscr{F}(\bar{x}_K)$,
- $M_{Cov}(\hat{K}) = \alpha_K^2 \mathcal{I}$.

Some important properties of $\mathscr{F}$

- Different choices for α_K:

$$\text{If } \alpha_K = \begin{cases} 1, \text{then the variance of } \hat{K} \text{ is 1 in every direction} \\ \left(\dfrac{\sqrt{\Pi_{j=1}^{d} \lambda_{K,j}}}{|K|} \right)^{1/d}, \text{ then } |\hat{K}| = 1. \end{cases}$$

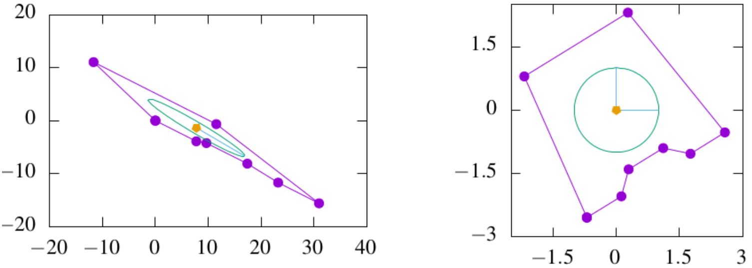

Figure: Demonstration of transformation (1), original element (left) and transformed element (right), α_K is such that $|\hat{K}| = 1$

Some further definition

Definition (regular and stable anisotropic mesh)

$\mathscr{K}_h$ is called a regular and stable anisotropic mesh if:

1. The reference configuration $\hat{K}$ for all $K \in \mathscr{K}_h$ is a regular and stable polytopal element according to Seminar 02.
2. Neighbouring elements behave similarly in their anisotropy, i.e., for two neighbouring elements K_1 and K_2, $\overline{K_1} \cap \overline{K_2} \neq \emptyset$, covariance matrices

$$M_{Cov}(K_1) = U_{K_1} \Lambda_{K_1} U_{K_1}^\top \qquad \text{and} \qquad M_{Cov}(K_2) = U_{K_2} \Lambda_{K_2} U_{K_2}^\top$$

we can write $\Lambda_{K_2} = (\mathcal{I} + \Delta^{K_1,K_2} \Lambda_{K_1})$ and $U_{K_2} = R^{K_1,K_2} U_{K_1}$ with $\Delta^{K_1,K_2} = diag\left(\delta_j^{K_1,K_2} : j = 1, \ldots, d\right)$, and a rotation matrix $R^{K_1,K_2} \in \mathbb{R}^{d \times d}$ such that for $j = 1, \ldots, d$ $0 \leq |\delta_j^{K_1,K_2}| < c_\delta < 1$ as well as $0 \leq \|R^{K_1,K_2} - \mathcal{I}\| \left(\frac{\lambda_{K_1}}{\lambda_{K_2}}\right)^{1/2} < c_R$ holds uniformly.

Mapping of patches

- Elements in regular and stable anisotropic meshes can be mapped onto a reference element
- For quasi-interpolant operators we need mapping properties of $\mathscr{F}$ for patches of elements

Theorem

Let $\mathscr{K}_h$ be a regular and stable anisotropic mesh, ω_z be the patch of elements corresponding to the node $z \in \mathscr{N}_h$,
and $K_1, K_2 \in \mathscr{K}_h$ with $K_1, K_2 \subset \omega_z$.
Then $\mathscr{F}_{K_1}(K_2)$ is regular and stable in the sense of the definition in Seminar 02 with slightly perturbed regularity and stability parameters $\tilde{\sigma}_{\mathscr{K}}$ and $\tilde{c}_{\mathscr{K}}$ depending only on the regularity and stability of $\mathscr{K}_h$.
Consequently, the mapped patch $\mathscr{F}_K(\omega_z)$ consists of regular and stable polytopal elements for all $K \in \mathscr{K}_h$ with $K \subset \omega_z$.

Mapping of patches

- The theorem has the consequences that
  1. the mapped patches $\mathscr{F}(\omega_K)$ and $\mathscr{F}(\omega_F)$ consist of regular and stable polytopal elements
  2. each node $z \in \mathscr{N}_h$ of a regular and stable anisotropic mesh belongs to a uniformly bounded number of elements and, vice versa, each element K has a uniformly bounded number of nodes on its boundary.
  3. for $K_1, K_2 \subset \omega_z$ we have for the mapped patch $\tilde{\omega} \in \{\mathscr{F}_{K_1}(\omega_z), \mathscr{F}_{K_1}(\omega_{K_1})\}$ and the neighboring elements that

$$h_{\tilde{\omega}} \leq c \text{ and } \frac{|K_2|}{|K_1|} \leq c$$

where $h_{\tilde{\omega}}$ is the diameter of the patch and c depends only on the regularity and stability parameters of the mesh.

Outline

Approximation space

- Previous definition of V_h yields harmonic functions $v_h \in V_h$ for each patch
- Now follow the more classical FEM approach
- Define $\hat{\Psi}$ on $\hat{K}$ as in Seminar 03
- Map $\hat{\Psi}$ onto K such that $\Psi^{ref} = \hat{\Psi} \circ \mathscr{F}_K$
- Basis functions Ψ^{ref} are not harmonic in K, but

$$div\left(\Lambda_K^{-1} \nabla \Psi^{ref}\right) = 0$$

- If $K = \hat{K}$ then the nodal basis functions Ψ_z^{ref} coincide with the former definition of the basis functions

Approximation space

- The approximation space is given by

$$V_h^{ref} = \{v \in H^1(\Omega) : div\left(\Lambda_K^{-1}\nabla v\right)|_K = 0 \text{ and } v|_{\partial K} \in \mathscr{P}^1_{pw}(\partial K) \forall K \in \mathscr{K}_h\}$$

- The space V_h^{ref} fulfills

$$\mathscr{P}^1(K) \subset V_h^{ref}|_K \text{ and } 0 \leq \Psi^{ref} \leq 1$$

Outline

Trace inequality

- Anisotropic trace inequality is extension of trace inequality for the isotropic regime

Theorem

Let $K \in \mathscr{K}_h$ be a polytoplal element of a regular and stable anisotropic mesh $\mathscr{K}_h$ with edge (d=2) or face (d=3) $F \in \mathscr{F}_h, F \subset \partial K$. It holds for $v \in H^1(K)$ that

$$\|v\|^2_{L_2(F)} \leq c\frac{|F|}{|K|}\left(\|v\|^2_{L_2(K)} + \|A_K^{-\top}\nabla v\|^2_{L_2(K)}\right)$$

where c depends only on the regularity and stability parameters of the mesh.

Best approximation result on patches

Theorem

Let $\mathscr{K}_h$ be a regular and stable anisotropic mesh with node $z \in \mathscr{N}_h$ and element $K \in \mathscr{K}_h$. Furthermore let ω_z and ω_K be the neighborhoods of z and K, respectively. $K \subset \omega_z$. For $\omega \in \{\omega_z, \omega_K\}$ with Π_ω, the L_2-projection over ω into the space of constants, it holds

$$\|v - \Pi_\omega v\|_{L_2(\omega)} \leq c\|A_K^{-\top} \nabla v\|_{L_2(\omega)},$$

$$\|v - \Pi_\omega v\|_{L_2(\omega)} \leq c \left(\sum_{K' \in \mathscr{K}_h : K' \subset \omega} \|A_{K'}^{-\top} \nabla v\|^2_{L_2(K')} \right)^{1/2},$$

for $v \in H^1(\omega)$, where c only depends on the regularity and stability of the mesh.

Outline

Quasi interpolators

- The general form of quasi interpolators for $v \in H^1(\Omega)$ is

$$\mathcal{J} v = \sum_{z \in \mathcal{N}_*} \left(\Pi_{\omega(\Psi_z)} v \right) \Psi_z \in V_h$$

where $\mathcal{N}_*$ and $\omega(\Psi_z)$ are chosen accordingly.

Note

Here we have V_h instead of V_h^{ref}, but the following results are valid for V_h^{ref}

Clément-type interpolation

- Denoted by $\mathcal{J}_C$
- The set of nodes is $\mathscr{N}_* = \mathscr{N}_h \backslash \mathscr{N}_{h,D}$
- The quantity $\omega(\Psi) = \omega_z$
- Following results are equivalent to results on anisotropic triangular meshes

Clément-type interpolation

Theorem

Let $\mathscr{K}_h$ be a regular and stable anisotropic mesh and $K \in \mathscr{K}_h$. The Clément interpolation operator fulfills for $v \in H^1_D(\Omega)$ the interpolation error estimate

$$\|v - \mathcal{J}_C v\|_{L_2(K)} \leq c\|A_K^{-\top} \nabla v\|_{L_2(\omega_K)}$$

and for an edge/face $\mathscr{F}(K) \backslash \mathscr{F}_{h,D}$

$$\|v - \mathcal{J}_C v\|_{L_2(F)} \leq c \frac{|F|^{1/2}}{|K|^{1/2}} \|A_K^{-\top} \nabla v\|_{L_2(\omega_F)}$$

where c only depends on the regularity and stability of the mesh.

Scott-Zhang-type interpolation

- Denoted by $\mathcal{J}_{SZ}$
- The set of nodes is $\mathcal{N}_* = \mathcal{N}_h$
- The quantity $\omega(\Psi) = F_z \in \mathcal{F}_h$ with $z \in \overline{F}_z$ and
  - $F_z \subset \Gamma_D$ if $z \in \overline{\Gamma}_D$
  - $F_z \subset \Omega \cup \Gamma_N$ if $z \in \Omega \cup \Gamma_N$

Lemma

Let $\mathcal{K}_h$ be a regular and stable anisotropic mesh and $K \in \mathcal{K}_h$. The Scott-Zhang interpolation operator fulfills for $v \in H^1(\Omega)$ the local stability

$$\|\mathcal{J}_{SZ} v\|_{L_2(K)} \leq c \left(\|v\|_{L_2(\omega_K)} + \|A_K^{-\top} \nabla v\|_{L_2(\omega_K)} \right)$$

where c only depends on the regularity and stability of the mesh.

Scott-Zhang-type interpolation

Lemma

Let $\mathscr{K}_h$ be a regular and stable anisotropic mesh and $K \in \mathscr{K}_h$. The Scott-Zhang interpolation operator fulfills for $v \in H^1(\Omega)$ the interpolation error estimate

$$\|v - \mathcal{J}_{SZ} v\|_{L_2(K)} \leq c \|A_K^{-\top} \nabla v\|_{L_2(\omega_K)}$$

where c only depends on the regularity and stability of the mesh.

Outline

Classical interpolation

- Classical interpolation valid for sufficiently regular functions, e.g., for $v \in H^2(\Omega)$
- Interpolation operator is given by

$$\mathcal{J}_h v = \sum_{z \in \mathcal{N}_h} v(z) \Psi_z^{ref} \in V_h^{ref}$$

- We have $\widehat{\mathcal{J}_h v} = \hat{\mathcal{J}}_h \hat{v}$

Classical interpolation

Lemma (Scaling of H^1-seminorm)

Let $K \in \mathscr{K}_h$ be a polytopal element of a regular and stable anisotropic mesh $\mathscr{K}_h$. For $v \in H^1(K)$ it is

$$\sqrt{\frac{\Pi_{j=2}^{d}\lambda_{K,j}}{\lambda_{K,1}}}|\hat{v}|^2_{H^1(\hat{K})} \leq |v|^2_{H^1(K)} \leq \sqrt{\frac{\Pi_{j=1}^{d-1}\lambda_{K,j}}{\lambda_{K,d}}}|\hat{v}|^2_{H^1(\hat{K})}$$

Classical interpolation

Theorem (Interpolation error)

Let $K \in \mathscr{K}_h$ be a polytopal element of a regular and stable anisotropic mesh $\mathscr{K}_h$. For $v \in H^2(\Omega)$ it is

$$|v - \mathcal{J}_h v|^2_{H^\ell(K)} \leq c\alpha_K^{-4} S_\ell(K) \sum_{i,j=1}^{d} \lambda_{K,i}\lambda_{K,j} L_K(u_{K,i}, u_{K,j}; v)$$

with

$$S_\ell(K) = \begin{cases} 1, & \text{for } \ell = 0, \\ \frac{1}{|K|}\sqrt{\frac{\Pi_{j=1}^{d-1}\lambda_{K,j}}{\lambda_{K,d}}}, & \text{for } \ell = 1 \end{cases}$$

where $L_K(u_{K,i}, u_{K,j}, v) = \int_K \left(u_{K,i}^\top H(v) u_{K,j}\right)^2 \, dx$ for $i, j = 1, \ldots, d$, $H(v)$ denotes the Hessian of v. The constant c depends only on the regularity and stability of the mesh.