Interpolation of non-smooth functions

Seminar on Numerical Analysis

Virtual Element Methods (VEM)

Andreas Schafelner

Johannes Kepler University, Linz

30 October 2018

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

FWF
Der Wissenschaftsfonds.

JKU

Outline

- Preliminaries
- Trace inequalities & best approximation
- Quasi-interpolation

- Preliminaries
- Trace inequalities & best approximation
- Quasi-interpolation

Non-smoothness

In S03: $u \in H^s(\Omega)$, $s \geq 2$.

Here: $u \in H^1(\Omega)$.

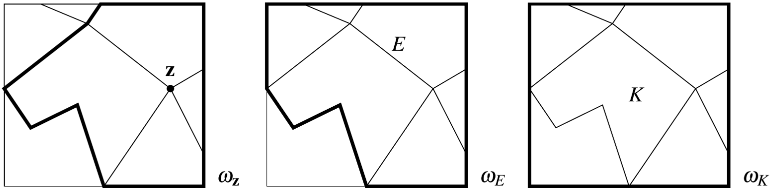
Definition (Neighbourhoods)

For each node $\mathbf{z}$ we consider the neighbourhood

$$\overline{\omega}_{\mathbf{z}} = \bigcup_{\mathbf{z} \in \mathcal{N}(K)} \overline{K},$$

and similarly, the neighbourhoods of edges, faces, and elements as

$$\overline{\omega}_E = \bigcup_{\mathbf{z} \in \mathcal{N}(E)} \overline{\omega}_{\mathbf{z}}, \qquad \overline{\omega}_F = \bigcup_{\mathbf{z} \in \mathcal{N}(F)} \overline{\omega}_{\mathbf{z}}, \qquad \overline{\omega}_K = \bigcup_{\mathbf{z} \in \mathcal{N}(K)} \overline{\omega}_{\mathbf{z}}.$$



Example of the neighbourhoods in 2D [1].

Lemma

Let $\mathcal{K}_h$ be regular and stable mesh of a two- or three-dimensional domain. Then, the following properties hold:

1. *Each element is covered by a uniformly bounded number of neighbourhoods of elements, i.e.* $|\{K' \in \mathcal{K}_h : K \subset \omega_{K'}\}| \leq c$, $\forall K \in \mathcal{K}_h$.
2. *For all* $\mathbf{z} \in \mathcal{N}_h$ *and* $K \subset \omega_{\mathbf{z}}$, *it is* $h_{\omega_{\mathbf{z}}} \leq c h_K$.

The constants c only depend on $\sigma_{\mathcal{K}}$, $\sigma_{\mathcal{F}}$ and $c_{\mathcal{K}}$.

Proof.

See [1]. □

Notation

F, $\mathcal{F}_h$, etc. means either the faces of the discretization (3D) or the edges (2D)

The approximation space in 3D

We use the first order approximation space

$$V_h = \{v \in H^1(\Omega) : \Delta v|_K = 0 \,\forall K \in \mathcal{K}_h \text{ and } v|_F \in V_h(F) \,\forall F \in \mathcal{F}_h\}.$$

- Preliminaries
- Trace inequalities & best approximation
- Quasi-interpolation

Lemma

Let $\mathcal{K}_h$ be a regular mesh, $v \in H^1(\Omega)$ for $K \in \mathcal{K}_h$ and $E \in \mathcal{E}(K)$. Then it holds

$$\|v\|_{L_2(E)} \leq c\left\{h_E^{-1/2}\|v\|_{L_2(T_E^{\mathrm{iso}})} + h_E^{1/2}|v|_{H^1(T_E^{\mathrm{iso}})}\right\}$$

with the isosceles triangle $T_E^{\mathrm{iso}} \subset K$ from S02, *where c only depends on the regularity parameter $\sigma_{\mathcal{K}}$.*

Lemma (Trace inequality)

Let $\mathcal{K}_h$ *be a regular and stable mesh,* $v \in H^1(\Omega)$ *for* $K \in \mathcal{K}_h$ *and* $F \in \mathcal{F}(K)$. *Then it holds*

$$\|v\|_{L_2(F)} \leq c \left(h_F^{-1/2} \|v\|_{L_2(K)} + h_F^{1/2} |v|_{H^1(K)} \right)$$

where c *only depends on the regularity parameter* $\sigma_{\mathcal{K}}$, $\sigma_{\mathcal{F}}$ *and* $c_{\mathcal{K}}$.

Proof.

[1] □

In S03, we used the Bramble-Hilbert polynomial on star-shaped domains.

Lemma

Let $\mathcal{K}_h$ be a regular and stable mesh and $k \in \mathbb{N}_0$. Then there exists for every function $v \in H^{k+1}(\omega)$ and every neighbourhood $\omega \in \{\omega_{\mathbf{z}}, \omega_F, \omega_K\}$ a polynomial $p \in \mathcal{P}^k(\omega)$ such that

$$|v - p|_{H^l(\omega)} \leq C h^{k+1-l} |v|_{H^{k+1}(\omega)} \quad \text{for } l = 0, \ldots, k+1,$$

where C only depends on $\sigma_{\mathcal{K}}$, $\sigma_{\mathcal{F}}$ and $c_{\mathcal{K}}$ as well as on k and the dimension d.

Definition (L_2-projection)

For $v \in H^1(\Omega)$ the L_2-projection is given by

$$\Pi_\omega v = \frac{1}{|\omega|} \int_\omega v(\mathbf{x}) \mathrm{d}\mathbf{x}$$

The Poincaré constant

We know

$$C_P(\omega) = \sup_{v \in H^1(\omega)} \frac{\|v - \Pi_\omega v\|_{L_2(\omega)}}{h_\omega |v|_{H^1(\omega)}} < \infty,$$

and depends on the shape of ω, e.g. for $d = 2$ and ω *convex*,

$$C_P(\omega) < 1/\pi.$$

Lemma

Let $\mathcal{K}_h$ be a regular and stable mesh. Then there exists a uniform constant c, which only depends on $\sigma_{\mathcal{K}}$, $\sigma_{\mathcal{F}}$ and $c_{\mathcal{K}}$, such that for every neighbourhood $\omega \in \{\omega_{\mathbf{z}}, \omega_F, \omega_K\}$ with $\mathbf{z} \in \mathcal{N}_h$, $F \in \mathcal{F}_h$ and $K \in \mathcal{K}_h$, it holds

$$\|v - \Pi_\omega v\|_{L_2(\omega)} \leq c h_\omega |v|_{H^1(\omega)}.$$

Proof.

Whiteboard: for $d = 2$ and $\omega = \omega_{\mathbf{z}}$ via the Poincaré constant.

Proof (cont.).

Admissible decomposition $\{\omega_i\}_{i=1}^n$ of ω with

$$\overline{\omega} = \bigcup_{i=1}^{n} \overline{\omega}_i,$$

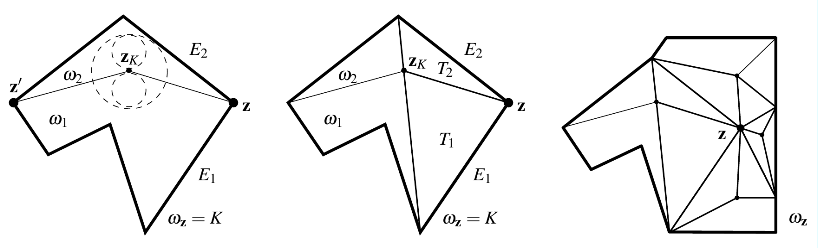
where ω_i are pairwise disjoint.
Then,

$$C_P(\omega) \leq \max_{1\leq i\leq n} \left\{ 8(n-1)\left(1 - \min_{1\leq j\leq n} \frac{|\omega_j|}{|\omega|}\right)\left(C_P^2(\omega_i) + 2C_P(\omega_i)\right) \frac{|\omega| h_{\omega_i}^2}{|T_i| h_\omega^2} \right\}$$

Proof (cont.).

We will need

$$|T_E| \geq \frac{1}{2} h_E \rho_K, \qquad h_K \leq c_{\mathcal{K}} h_E, \qquad \frac{h_K}{\rho_K} \leq \sigma_{\mathcal{K}}.$$



Construction of admissible decomposition for K and $\omega_{\mathbf{z}}$

□

Remark.

The general case yields

$$C_P(\boldsymbol{\omega}_{\mathbf{z}}) \leq \left(16(n-1)c_{\mathcal{K}}\boldsymbol{\sigma}_{\mathcal{K}} \max_{1\leq i\leq n}\left\{C_P^2(\boldsymbol{\omega}_i) + 2C_P(\boldsymbol{\omega}_i)\right\}\right)^{1/2}$$

- Preliminaries
- Trace inequalities & best approximation
- Quasi-interpolation

Goal

- define interpolation for general functions in $H^1(\Omega)$

In general: no embedding into C^0 $\Rightarrow$ no Lagrange-interpolation operators

Solution.

Construct quasi-interpolation operators.

Definition (Quasi-interpolant)

For $v \in H^1(\Omega)$ we define the quasi-interpolation operator

$$\mathfrak{I}v = \sum_{\mathbf{z} \in \mathcal{N}_*} (\Pi_{\omega(\psi_{\mathbf{z}})} v)\psi_{\mathbf{z}} \in V_h,$$

where the set of nodes $\mathcal{N}_*$ and the neighbourhoods $\omega(\psi_{\mathbf{z}})$ have to be specified.

Clément-type interpolation

Choice of $\mathcal{N}_*$ & $\omega(\psi_{\mathbf{z}})$

We choose

$$\mathcal{N}_* = \mathcal{N}_h \setminus \mathcal{N}_{h,D},$$

i.e. all nodes which do not lie on the Dirichlet boundary, and

$$\boldsymbol{\omega}(\psi_{\mathbf{z}}) = \boldsymbol{\omega}_{\mathbf{z}},$$

i.e. the neighbourhood of the nodes.

Theorem

Let $\mathcal{K}_h$ be a regular and stable mesh and let $F \in \mathcal{F}_h$ and $K \in \mathcal{K}_h$. The Clément interpolation operator fulfils for $v \in H^1_D(\Omega)$ the interpolation error estimates

$$\|v - \mathfrak{I}_C v\|_{L_2(K)} \leq c h_K |v|_{H^1(\omega_K)},$$

and

$$\|v - \mathfrak{I}_C v\|_{L_2(F)} \leq c h_F^{1/2} |v|_{H^1(\omega_F)},$$

where the constants c only depend on $\sigma_{\mathcal{K}}$, $\sigma_{\mathcal{F}}$ and $c_{\mathcal{K}}$.

Roadmap of Proof.

- first estimate
  - partition of unity property on K
  - 2 cases: $\mathbf{z} \in \mathcal{N}_h \setminus \mathcal{N}_{h,D}$ and $\mathbf{z} \in \mathcal{N}_{h,D}$
  - we need

$$h_{K'}^{d-1} \leq c|F'|, \qquad h_{F'} \leq h_{K'}, \qquad h_{\omega_{\mathbf{z}}} \leq c h_{K'} \leq c h_{F'}.$$

- second estimate
  - similarly
  - we need

$$|F| \leq h_F^{d-1}, \qquad h_{K'}^{1-d/2} \leq h_{F'}^{1-d/2}.$$

Roadmap of Proof.

- first estimate
  - partition of unity property on K
  - 2 cases: $\mathbf{z} \in \mathcal{N}_h \setminus \mathcal{N}_{h,D}$ and $\mathbf{z} \in \mathcal{N}_{h,D}$
  - we need

$$h_{K'}^{d-1} \leq c|F'|, \qquad h_{F'} \leq h_{K'}, \qquad h_{\omega_{\mathbf{z}}} \leq c h_{K'} \leq c h_{F'}.$$

- second estimate
  - similarly
  - we need

$$|F| \leq h_F^{d-1}, \qquad h_{K'}^{1-d/2} \leq h_{F'}^{1-d/2}.$$

□

Scott-Zhang-type interpolation

Choice of $\mathcal{N}_*$ & $\omega(\psi_z)$

We choose

$$\mathcal{N}_* = \mathcal{N}_h,$$

and

$$\omega(\psi_{\mathbf{z}}) = F_{\mathbf{z}},$$

where $F_{\mathbf{z}} \in \mathcal{F}_h$ is an edge/face with $\mathbf{z} \in \overline{F}_{\mathbf{z}}$ and

$$F_{\mathbf{z}} \subset \Gamma_D \text{ if } \mathbf{z} \in \overline{\Gamma}_D \qquad \text{and} \qquad F_{\mathbf{z}} \subset \Omega \cup \Gamma_N \text{ if } \mathbf{z} \in \Omega \cup \overline{\Gamma}_N.$$

Lemma

Let $\mathcal{K}_h$ be a regular and stable mesh and $K \in \mathcal{K}_h$. The Scott-Zhang interpolation operator fulfils for $v \in H^1(\Omega)$ the local stability

$$\|\mathfrak{I}_{SZ} v\|_{L_2(K)} \leq c \left(\|v\|_{L_2(\omega_K)} + h_K |v|_{H^1(\omega_K)} \right),$$

where the constants c only depend on $\sigma_{\mathcal{K}}$, $\sigma_{\mathcal{F}}$ and $c_{\mathcal{K}}$.

Proof.

See [1]. □

Theorem

Let $\mathcal{K}_h$ be a regular and stable mesh and $K \in \mathcal{K}_h$. The Scott-Zhang interpolation operator fulfils for $v \in H^1(\Omega)$ the interpolation error estimate

$$\|v - \mathfrak{I}_{SZ}v\|_{L_2(K)} \leq c h_K |v|_{H^1(\omega_K)},$$

where the constants c only depend on $\sigma_{\mathcal{K}}$, $\sigma_{\mathcal{F}}$ and $c_{\mathcal{K}}$.

Proof.

See [1]. □

[1] Weisser, S. BEM-based Finite Element Approaches on Polytopal Meshes. *Manuscript*.

Thank you!