Virtual Element Methods for plate bending problems

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Limitations

Limitations of this paper

- Limited to **CONVEX and POLYGONAL** domains (= Regularity assumption and Babuska-Paradoxon)
- **NO numerical integration** is used
- **NO robustness** is shown
- The issue of **LOCKING phenomena** is not considered

Advantages of this paper

- The VEM technique admits **NON-POLYNOMIAL function representation** without explicit knowledge

The Continuous problem

- $\Omega \subset \mathbb{R}^2$ convex polygonal domain
- $\Gamma := \partial\Omega$
- $f \in L^2(\Omega)$ transversal load

The Kirchhoff-Love model for a clamped plate

For $D := Et^3/12(1-\nu^2)$ consider

$$D\Delta^2 w = f \text{ in } \Omega \text{ with } w = \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma$$

Variational formulation

Find $w \in V := H_0^2(\Omega)$ such that

$$a(w,v) = (f,v) \qquad \forall v \in H_0^2(\Omega)$$
$$a(w,v) = D(1-\nu)\sum_{i,j}(w_{,ij}, v_{,ij}) + D\nu(\Delta w, \Delta v)$$

Existence, uniqueness, continuous dependence

Boundary conditions and Friedrich inequality imply

$$a(u,v) \leq M\|u\|_V\|v\|_V \qquad u,v \in V$$
$$a(v,v) = \|v\|_a^2 \geq \alpha\|v\|_V^2 \qquad v \in V$$

Hence **well-posedness**

$$\|w\|_V \leq C\|f\|_0$$

Notation

- $\mathcal{D} \subset \mathbb{R}^2$ domain
- $\boldsymbol{n} = (n_1, n_2)$ outwart unit normal vector to $\partial\mathcal{D}$
- $\boldsymbol{t} = (t_1, t_2)$ counterclockwise unit tangent vector to $\partial\mathcal{D}$
- **Moment tensor** for $v \in H^2(\Omega)$, $\boldsymbol{M} = \Big(M_{ij}(v)\Big)_{i,j=1}^{2}$

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} v_{,11} \\ v_{,22} \\ v_{,12} \end{bmatrix}$$

- $\boldsymbol{M}_n := \sum_j M_{ij} n_j$
- $M_{\boldsymbol{nn}}(v) := \sum_{i,j} M_{ij} n_i n_j$ normal bending moment
- $M_{\boldsymbol{nt}}(v) := \sum_{i,j} M_{ij} n_i t_j$ normal twisting moment
- $Q_{\boldsymbol{n}}(v) := \sum_{i,j} M_{ij,i} n_j$ normal shear force

Notation

We have

$$
\begin{aligned}
a^{\mathcal{D}}(w,v) =& D(1-\nu)\sum_{i,j}(w_{,ij}, v_{,ij})_{\mathcal{D}} + D\nu(\Delta w, \Delta v)_{\mathcal{D}} \\
\overset{IP^2}{=}& \int_{\mathcal{D}} D\Delta^2 wv dx + \int_{\partial\mathcal{D}} M_{\boldsymbol{nn}}(w)\frac{\partial v}{\partial n} dt \\
&- \int_{\partial\mathcal{D}} \Big(Q_{\boldsymbol{n}}(w) + \frac{\partial M_{\boldsymbol{nt}}(w)}{\partial t}\Big) v dt
\end{aligned}
$$

The discrete problem

- $\{\mathcal{T}_h\}_h$ decomposition of Ω into elements K
- $\mathcal{E}_h$ edges e of $\mathcal{T}_h$

Assumption **H0**

$\exists N \in \mathbb{N}, \gamma > 0 \forall h > 0, K \in \mathcal{T}_h :$

- $\#\mathcal{E}(K) \leq N$
- $\frac{\min_{e \in \mathcal{E}(K)} |e|}{h_K} \geq \gamma$
- K is **star-shaped** wrt. a ball of radius γh_K

The discrete problem

Assumption **H1**

$\forall h > 0$ we are given:

- $V_h \subset V$ ($V_h^K := V_h|_K$)
- **Symmetric bilinear form** $a_h : V_h \times V_h \to \mathbb{R}$ with

$$a_h(u_h, v_h) = \sum_K a_h^K(u_h, v_h) \qquad \forall u_h, v_h \in V_h$$

  where a_h^K is symmetric on $V_h^K \times V_h^K$
- $f_h \in V_h'$

The discrete problem

Assumption **H2**

$\exists k \geq 2 \forall h > 0, K \in \mathcal{T}_h$:

- k-consistency: $\forall p \in \mathcal{P}_k, v_h \in V_h : a_h^K(p, v_h) = a^K(p, v_h)$
- stability:
  $\exists \alpha_*, \alpha^* > 0 : \alpha_* a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* a^K(v_h, v_h) \qquad \forall v_h \in V_h$

The discrete problem

$$a(u,v) = \sum_K a^K(u,v) \qquad \forall u,v \in V$$

$$\|v\|_V = \Big(\sum_K |v|^2_{V,K}\Big)^{1/2} \qquad \forall v \in V$$

$$|v|_{h,V} := \Big(\sum_K |v|^2_{V,K}\Big)^{1/2} \qquad \forall v \in \prod_K H^2(K)$$

An abstract convergence theorem

- Symmetry of a_h and continuity of a^K imply continuity of a_h

$$
\begin{aligned}
a_h^K(u,v) \leq & \Big(a_h^K(u,u)\Big)^{1/2}\Big(a_h^K(v,v)\Big)^{1/2} \\
\leq & \alpha^* \Big(a^K(u,u)\Big)^{1/2}\Big(a^K(v,v)\Big)^{1/2} \\
\leq & \alpha^* M \|u\|_{V,K}\|v\|_{V,K} \qquad \forall u,v \in V_h
\end{aligned}
$$

- **Convergence result** for the discrete problem

An abstract convergence theorem

Theorem

H1, **H2** $\Longrightarrow$

$$\textit{Find } w_h \in V_h : a_h(w_h, v_h) = \langle f_h, v_h \rangle \qquad \forall v_h \in V_h$$

has a unique solution w_h.
Moreover: $\forall w_I \in V_h, w_\pi \in$ *piecewise* $\mathbb{P}_k$*:*

$$\|w - w_h\|_V \leq C\Big(\|w - w_I\|_V + \|w - w_\pi\|_{h,V} + \|f - f_h\|_{V_h'}\Big)$$

with $C = C(\alpha, \alpha_*, \alpha^*, M) > 0$.

Construction of V_h, a_h, f_h

- Want to satisfy **H1**, **H2**.
- Degree of accuracy $k \geq 2$: Introduce auxiliary quantities

$$r = \max\{3, k\} \qquad s = k - 1 \qquad m = k - 4$$

  which will be related to
  - the polynomial degree in V_h
  - the polynomial degree of their normal derivative on each edge
  - the DOFs internal to each element.
- For each $K \in \mathcal{T}_h$

$$\begin{aligned} V_h^K := \{v \in H^2(K) :& \Delta^2 v \in \mathbb{P}_m(K), \\ & v|_e \in \mathbb{P}_r(e), (v_{,\boldsymbol{n}})|_e \in \mathbb{P}_s(e), \\ & \forall e \in \partial K\} \end{aligned}$$

Construction of V_h, a_h, f_h

- For $t \in \mathbb{N}$ and edge $e \in \mathcal{E}_h$ with midpoint $\boldsymbol{x}_e$ intorduce normalized monomials

$$\mathcal{M}_t^e := \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_e}{h_e} \right)^{\boldsymbol{\beta}}, |\boldsymbol{\beta}| \leq t \right\}$$

- For $t \in \mathbb{N}$ and element $K \in \mathbb{T}_h$ with barycenter $\boldsymbol{x}_k$ introduce normalized monomials

$$\mathcal{M}_t^K := \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_K}{h_K} \right)^{\boldsymbol{\beta}}, |\boldsymbol{\beta}| \leq t \right\}$$

LOCAL Degrees of Freedom

DOFs for element K

1. $\forall \xi \in \mathcal{N}(K) : v(\xi)$
2. $\forall \xi \in \mathcal{N}(K) : h_\xi \nabla v(\xi)$
3. If $r > 3$: $\forall e \in \mathcal{E}(K) : \frac{1}{h_e} \int_e q(\xi) v(\xi) d\xi, \forall q \in \mathcal{M}^e_{r-4}$
4. If $s > 1$: $\forall e \in \mathcal{E}(K) : \int_e q(\xi) \frac{\partial v}{\partial \boldsymbol{n}} d\xi, \forall q \in \mathcal{M}^e_{s-2}$
5. If $m \geq 0$: $\frac{1}{h_K^2} \int_K q(x) v(x) dx, \forall q \in \mathcal{M}^K_m$

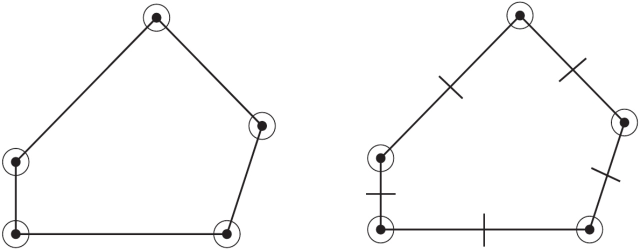

Fig. 1. Local d.o.f. for the lowest-order element: $k = 2$ (left), and next to the lowest $k = 3$ (right).

LOCAL Degrees of Freedom

Let $P_m^K v : L^2(K) \to \mathbb{P}_m(K), m \geq 0$ be the $L^2(K)$-projector onto $\mathbb{P}_m(K)$

Proposition

1. In each element K the DOFs 1–3 uniquely determine a polynomial of degree $\leq r$ on each edge of K.
2. The DOFs 2–4 uniquely determine a polynomial of degree $\leq s$ on each edge of K.
3. DOF 5 is equivalent to prescribing $P_m^K v$ in K.

Proposition

The above DOFs are **UNISOLVENT** in V_h^K.

Proof.

In the same spirit as the 2nd order elliptic case. □

GLOBAL Degrees of Freedom

Global construction of V_h

$$V_h = \{v \in V : v|_e \in \mathbb{P}_r(e), \frac{\partial v}{\partial \boldsymbol{n}|_e}, \Delta^2 v|_K \in \mathbb{P}_m(K), \forall e \in \mathcal{E}_h, K \in \mathcal{T}_h\}$$

with DOFs over **INTERNAL vertices/edges**.

GLOBAL Degrees of Freedom

Theorem

Let the DOFs of V_h be given by $g_1, g_2, \ldots, g_{\mathcal{G}}$. Then for every smooth enough w there exists a unique $w_I \in V_h$ such that

$$g_i(w - w_I) = 0 \qquad \forall i = 1, 2, \ldots, \mathcal{G}$$

Furthermore, for $\alpha, \beta \in \mathbb{N}$ one has

$$\|w - w_i\|_{\alpha,\Omega} \leq Ch^{\beta-\alpha}|w|_{\beta,\Omega} \qquad \alpha = 0, 1, 2 \qquad \alpha \leq \beta \leq k + 1$$

with C independent of h.

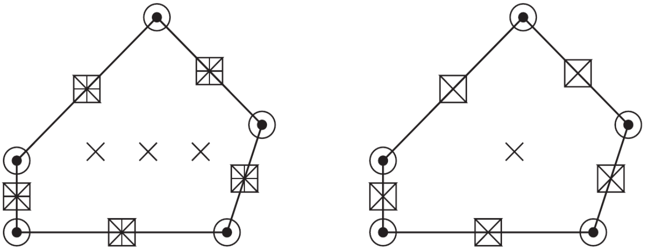

Fig. 2. Local d.o.f. for the (5,4,1) element with $k = 5$ (left), and for the (5,3,0) element with $k = 4$ (right).

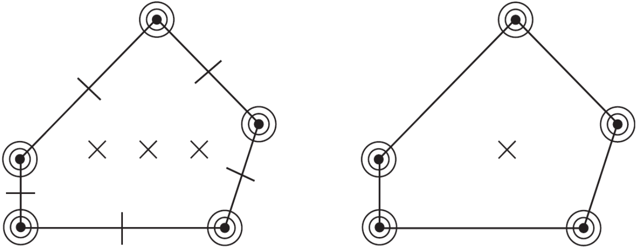

Fig. 3. Alternative d.o.f. for the elements of Fig. 2. We have an "Argyris-like" element on the left and a "Bell-like" element on the right.

Construction of a_h

AS IN THE PREVIOUS SEMINARS: Want to construct a_h according to assumptions (stability and consistency).

- Per construction: $a^K(p, v), p \in \mathbb{P}_k(K), v \in V_h^K$ can be computed ✓
- We have

$$a^K(p, v) = D \int_K \underbrace{\Delta^2 p}_{\in \mathbb{P}_{k-4}(K)} \; v dx + \int_{\partial K} \underbrace{M_{\boldsymbol{nn}}(p)}_{\in \mathbb{P}_{k-2}(e)} \frac{\partial v}{\partial \boldsymbol{n}} dt$$
$$+ \int_{\partial K} \Big(\underbrace{Q_{\boldsymbol{n}}(p) + \frac{\partial M_{\boldsymbol{nt}(p)}}{\partial t}}_{\in \mathbb{P}_{k-3}(e)} \Big) v dt$$

Can be computed **WITHOUT KNOWING polynomial v INSIDE**!

Construction of a_h

- Introduce quasi-average $\widehat{\varphi}$ of $\varphi \in C^0(\overline{K})$

$$\widehat{\varphi} := \frac{1}{l} \sum_{i=1}^{l} \varphi(\boldsymbol{x}^i)$$

  with vertices $\boldsymbol{x}^i, i = 1, 2, \ldots, l$ of K.
- Introduce $\Pi_k^K : V_h^K \to \mathbb{P}_k(K) \subset V_h^K$ via

$$a^K(\Pi_k^K \psi, q) = a^K(\psi, q) \qquad \forall \psi \in V_h^K, q \in \mathbb{P}_k(K)$$
$$\widehat{\Pi_k^K \psi} = \widehat{\psi} \qquad \widehat{\nabla \Pi_k^K \psi} = \widehat{\nabla \psi}$$

  1st line $\implies$ For $v \in \mathbb{P}_k(K)$ $(\Pi_k^K v)_{,ij} = v_{,ij}$ for $i, j = 1, 2$
  +2nd line $\implies$ $\Pi_k^K v = v$ for $\forall v \in \mathbb{P}_k(K)$

Construction of a_h

- **AS IN THE PREVIOUS SEMINARS:** Choice $a_h^K(u,v) = a^K(\Pi_k^K u, \Pi_k^K v)$ would be consistent, but NOT stable $\Longrightarrow$ need to add stabilizing term
- $S^K(u,v)$ needs to be symmetric, positive definite with

$$c_0 a^K(v,v) \leq S^K(v,v) \leq c_1 a^K(v,v), \qquad \forall v \in V_h^K, \Pi_k^K v = 0$$

for $c_0, c_1 > 0$ independent of K, h_K.

Local contributions for a_h

$$a_h^K(u,v) := a^K(\Pi_k^K u, \Pi_k^K v) + S^K(u - \Pi_k^K u, v - \Pi_k^K v)$$

Proposition

The above bilinear form satisfiecs the **assumptions of STABILITY and CONSISTENCY**.

Proof.

Previous seminars. □

Choice of S^K

- Choice of S^K has to depend on $a(.,.)$ and on $g_1, g_2, \dots, g_{\mathcal{G}}$
- Choose

$$S^K(v,w) = D \sum_{i=1}^{\mathcal{G}} g_i(v) g_i(w) h_i^{-2}$$

Construction of f_h: The EASY way

Assume $k \geq 4$ (ie. every element has at least ONE INTERNAL DOF)

- Define f_h on each element K as the $L^2(K)$-projection of f on piecewise polynomials of degree $m = k - 4$

$$f_h = P^K_{k-4} f \qquad \forall K \in \mathcal{T}_h$$

  Hence

$$\langle f_h, v_h \rangle = \sum_{K \in \mathcal{T}_h} \int_K f_h v_h dx = \sum_{K \in \mathcal{T}_h} \int_K (P^K_{k-4} f) v_h dx$$
$$= \int_{K \in \mathcal{T}_h} \int_K f (P^K_{k-4} v_h) dx$$

  which can be exactly computed by using internal DOFs.

- One has $\|f - f_h\|_{V_h'} \leq C h^{k-1} \Big(\sum_{K \in \mathcal{T}_h} |f|^2_{k-3,K} \Big)^{1/2}$

Construction of f_h: The NOTSOEASY way

Case studies for $k = 2, 3, 4$

$$\|f - f_h\|_{V_h'} \leq Ch^{k-1}\Big(\sum_{K\in\mathcal{T}_h} \|f\|_{0,K}^2\Big)^{1/2}$$

$$\|f - f_h\|_{V_h'} \leq Ch^{k-1}\Big(\sum_{K\in\mathcal{T}_h} |f|_{1,K}^2\Big)^{1/2}$$