The Hitchhikers Guide to VEM

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Overview

- Last time in the NUMA-Seminar
- Stiffness matrix revisited
- Yet another projection
- Mass matrix

Last Time in the NUMA-Seminar

- VEM for Poisson problem
- Very general decomposition

$$V^{K,k} := \{v \in H^1(K) : v_{|\partial K} \in \mathbb{B}_k(\partial K), \Delta v_{|K} \in \mathbb{P}_{k-2}(K)\}$$

- $\mathcal{V}^{K,k}$: v_h at **vertices**
- $\mathcal{E}^{K,k}$: v_h at $k-1$ uniformly spaced points on **edges** e
- $\mathcal{P}^{K,k}$: **moments** $\frac{1}{|K|}\int_K m(x)v_h(x)dx \quad \forall m \in \mathcal{M}_{k-2}(K)$

Assembling: Stiffness Matrix Revisited

- $\Pi_K^\nabla : V^{K,k} \to \mathbb{P}_k(K) \subset V^{K,k}$

$$\begin{cases} a^K(\Pi_K^\nabla v_h, q) = a^K(v_h, q_k) & \forall q_k \in \mathbb{P}_k(K) \\ P_0 \Pi_K^\nabla v_h = P_0 v_h \end{cases} \tag{1}$$

- $\mathcal{M}_k$ spans $\mathbb{P}_k$

$$\Pi_K^\nabla v_h = \sum_{\beta=1}^{n_k} s^\beta m_\beta$$

- Rewrite (1)

$$\sum_{\beta=1}^{n_k} s^\beta \left(\nabla m_\alpha, \nabla m_\beta\right)_{0,K} = \left(\nabla m_\alpha, \nabla v_h\right) \tag{2}$$

$$\sum_{\beta=1}^{n_k} s^\beta P_0 m_\beta = P_0 v_h \tag{3}$$

Assembling: Stiffness Matrix Revisited

- Linear system for s^β

$$\begin{bmatrix} P_0 m_1 & P_0 m_2 & \dots & P_0 m_{n_k} \\ 0 & (\nabla m_2, \nabla m_2) & \dots & \left(\nabla m_2, \nabla m_{n_k}\right) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \left(\nabla m_{n_k}, \nabla m_2\right) & \dots & \left(\nabla m_{n_k}, \nabla m_{n_k}\right) \end{bmatrix} \begin{bmatrix} s^1 \\ s^2 \\ \vdots \\ s^{n_k} \end{bmatrix} = \begin{bmatrix} P_0 v_h \\ (\nabla m_2, \nabla v_h) \\ \vdots \\ \left(\nabla m_{n_k}, \nabla v_h\right) \end{bmatrix} \tag{4}$$

- $\mathbf{G}\underline{s} = \underline{b}$

✓ $\mathbf{G}$ computable

✓ $\underline{b}$ computable (shown last time)

Assembling: Stiffness Matrix Revisited

- Applied to basis functions φ_i

$$\Pi_K^\nabla \varphi_i = \sum_{\alpha=1}^{n_k} s_i^\alpha m_\alpha$$

- $\underline{s}^{(i)} = G^{-1}\underline{b}^{(i)}$

$$\mathbf{B} := \begin{bmatrix} \underline{b}^{(1)} & \underline{b}^{(2)} & \dots & \underline{b}^{(N^{dof})} \end{bmatrix} \tag{5}$$

- Matrix representation $\mathbf{\Pi}_*^\nabla$ of $\Pi^\nabla : V_k(K) \to \mathbb{P}_k(K)$ in the basis $\mathcal{M}_k$

$$\mathbf{\Pi}_*^\nabla = \mathbf{G}^{-1}\mathbf{B}$$

Assembling: Stiffness Matrix Revisited

- Canonical basis

$$\Pi^{\nabla}\varphi_i = \sum_{j=1}^{N^{dof}} \pi_i^j \varphi_j$$

with $\pi_i^j = dof_j(\Pi^{\nabla}\varphi_i)$

$$\Pi^{\nabla}\varphi_i = \sum_{\alpha=1}^{n_k} s_i^{\alpha} m_{\alpha} = \sum_{\alpha=1}^{n_k} s_i^{\alpha} \sum_{j=1}^{N^{dof}} dof_j(m_{\alpha})\varphi_j$$

- $\pi_i^j = \sum_{\alpha=1}^{n^k} s_i^{\alpha} dof_j(m_{\alpha})$

Assembling: Stiffness Matrix Revisited

- $\mathbf{D}_{i\alpha} := dof_i(m_\alpha)$
- Hence:

$$\pi_i^j = \sum_{\alpha=1}^{n_k} (G^{-1}B)_{\alpha i} D_{j\alpha} = (DG^{-1}B)_{ji}$$

- Matrix representation $\mathbf{\Pi}^\nabla$ of $\Pi^\nabla : V_k(K) \to V_k(K)$ in the canonical basis

$$\mathbf{\Pi}^\nabla = \mathbf{D}\mathbf{G}^{-1}\mathbf{B} = \mathbf{D}\mathbf{\Pi}_*^\nabla$$

- Can show $\mathbf{G} = \mathbf{B}\mathbf{D}$

Assembling: Stiffness Matrix Revisited

- Split $\varphi_i = \Pi^\nabla \varphi_i + (I - \Pi^\nabla)\varphi_i$
- Obtain:

$$(K_K)_{ij} = \left(\nabla \Pi^\nabla \varphi_i, \nabla \Pi^\nabla \varphi_j\right) + \left((I - \Pi^\nabla)\varphi_i, (I - \Pi^\nabla)\varphi_j\right)$$

- Approximate:

$$\left((I - \Pi^\nabla)\varphi_i, (I - \Pi^\nabla)\varphi_j\right) \approx \sum_{r=1}^{N^{dof}} dof_r(\varphi_i - \Pi_k^K \varphi_i) dof_r(\varphi_j - \Pi_k^K \varphi_j)$$

Assembling: Stiffness Matrix Revisited

- Obtain:

$$\mathbf{K}_\mathbf{K}^\mathbf{h} = (\mathbf{\Pi}_*^\nabla)^T \tilde{\mathbf{G}} \mathbf{\Pi}_*^\nabla + (\mathbf{I} - \mathbf{\Pi}^\nabla)^T (\mathbf{I} - \mathbf{\Pi}^\nabla)$$

- $\tilde{\mathbf{G}} = \mathbf{G}$ with first row set to 0
- $\mathbf{B}, \mathbf{D}, \mathbf{G}$ depend only on the shape of the polygon K and not on its size
- $\mathbf{K}_\mathbf{K}^\mathbf{h}$ is (in general) not close to $\mathbf{K}_\mathbf{K}$

Assembling: Stiffness Matrix Revisited

Summary (Stiffness matrix)

- Compute $\mathbf{B}, \mathbf{D}, \mathbf{G}$
- Compute $\mathbf{\Pi}_*^{\nabla} = \mathbf{G}^{-1}\mathbf{B}$ and $\mathbf{\Pi}^{\nabla} = \mathbf{D}\mathbf{\Pi}_*^{\nabla}$
- Compute $\mathbf{K}_{\mathbf{K}}^{\mathbf{h}} = (\mathbf{\Pi}_*^{\nabla})^T \tilde{\mathbf{G}} \mathbf{\Pi}_*^{\nabla} + (\mathbf{I} - \mathbf{\Pi}^{\nabla})^T (\mathbf{I} - \mathbf{\Pi}^{\nabla})$

Assembling: Mass Matrix

- $\mathbf{M_K}_{ij} := \int_K \varphi_i \varphi_j \mathrm{dx}$
- Not possible with the current dofs

$\rightarrow$ L^2-projector

Assembling: Mass Matrix

- Split $\varphi_i = \Pi^0\varphi_i + (I - \Pi^0)\varphi_i$
- Obtain:

$$(M_K)_{ij} = \left(\Pi^0\varphi_i, \Pi^0\varphi_j\right) + \left((I - \Pi^0)\varphi_i, (I - \Pi^0)\varphi_j\right)$$

- Approximate:

$$\left((I - \Pi^0)\varphi_i, (I - \Pi^0)\varphi_j\right) \approx |K| \sum_{r=1}^{N^{dof}} dof_r(\varphi_i - \Pi^0\varphi_i) dof_r(\varphi_j - \Pi^0\varphi_j)$$

$$\mathbf{M_K^h} = \mathbf{C}^T\mathbf{H}^{-1}\mathbf{C} + |K|(\mathbf{I} - \mathbf{\Pi^0})^T(\mathbf{I} - \mathbf{\Pi^0})$$

Yet Another Projection

- L^2-projection: Find $\Pi^0 v_h \in \mathbb{P}_k(K)$ such that

$$(p_k, \Pi^0 v_h) = (p_k, v_h) \quad \forall p_k \in \mathbb{P}_k(K)$$

- $\mathbf{H}_{\alpha\beta} := (m_\alpha, m_\beta)$
- $\Pi^0 v_h = \sum_{\alpha=1}^{n_k} t^\alpha m_\alpha$
- $c^\alpha := (m_\alpha, v_h)$

- Similar to before

$$\mathbf{H}\underline{t} = \underline{c}$$

Yet Another Projection

- Similar to before

$$\underline{t} = \mathbf{H}^{-1}\underline{c}$$

✓ $\mathbf{H}$ computable

✗ $\underline{c}$ computable (no moments for $|\alpha| = k - 1, k$)

Yet Another Projection

- Can compute $\Pi^\nabla v_h$
- For $v_h \in V_k(K)$: $v_h \sim \Pi^\nabla v_h$ and $v_h \sim \Pi^0 v_h$
- Hence: $\Pi^0 \sim \Pi^\nabla$

Idea Replace $c^\alpha := (m_\alpha, v_h)$ for $|\alpha| = k-1, k$ by

$$c^\alpha := (m_\alpha, \Pi^\nabla v_h)$$

✓ $\underline{c}$ computable

? Error ?

A Virtual Space for Virtual Elements

Introduce new space $W_k(K)$ with the good properties of $V_k(K)$

- polynomials on edges
- $\mathbb{P}_k(K) \subset W_k(K)$
- same dofs

Additionally:

- For $|\alpha| = k-1, k$ and $w_h \in W_k(K)$

$$\int_K w_h m_\alpha = \int_K \Pi^\nabla w_h \ m_\alpha$$

A Virtual Space for Virtual Elements

- if space exists, previous computations hold unchanged
- $dofs(v_h) = dofs(w_h) \implies \Pi^\nabla v_h = \Pi^\nabla w_h$

Summary (L^2-projector)

- Compute $\mathbf{H}$
- Compute $\Pi^\nabla v_h \equiv \Pi^\nabla w_h$ using only dofs
- Compute $\underline{c}$
- Solve $\underline{t} = \mathbf{H}^{-1}\underline{c}$
- Compute $\Pi^0 w_h = \sum_{\alpha=1}^{n_k} t^\alpha m_\alpha = \sum_{\alpha=1}^{n_k} t^\alpha \sum_{i=1}^{N^{dof}} D_{\alpha i}\varphi_i$

Matrix Representation

- Monomial basis:

$$(\mathbf{\Pi}^{\mathbf{0}}_{*})_{\alpha i} := t^{\alpha}(\varphi_i) = (\mathbf{H}^{-1}\mathbf{C})_{\alpha i}$$

with

$$\mathbf{C}_{\alpha i} := (m_\alpha, \varphi_i) = \begin{cases} (m_\alpha, \varphi_i) & 1 \leq \alpha \leq n_{k-2} \\ (m_\alpha, \Pi^{\nabla}\varphi_i) & n_{k-2}+1 \leq \alpha \leq n_k \end{cases}$$

- VEM-basis:

$$(\mathbf{\Pi}^{\mathbf{0}})_{ji} := (\mathbf{D}\mathbf{H}^{-1}\mathbf{C})_{ji}$$

Assembling: Mass Matrix

- Split $\varphi_i = \Pi^0\varphi_i + (I - \Pi^0)\varphi_i$
- Obtain:

$$(M_K)_{ij} = \left(\Pi^0\varphi_i, \Pi^0\varphi_j\right) + \left((I - \Pi^0)\varphi_i, (I - \Pi^0)\varphi_j\right)$$

- Approximate:

$$\left((I - \Pi^0)\varphi_i, (I - \Pi^0)\varphi_j\right) \approx |K| \sum_{r=1}^{N^{dof}} dof_r(\varphi_i - \Pi^0\varphi_i) dof_r(\varphi_j - \Pi^0\varphi_j)$$

$$\mathbf{M_K^h} = \mathbf{C}^T\mathbf{H}^{-1}\mathbf{C} + |K|(\mathbf{I} - \mathbf{\Pi^0})^T(\mathbf{I} - \mathbf{\Pi^0})$$

Summary

$$\mathbf{M_K^h} = \mathbf{C}^T \mathbf{H}^{-1} \mathbf{C} + |K| (\mathbf{I} - \mathbf{\Pi^0})^T (\mathbf{I} - \mathbf{\Pi^0})$$

$$\mathbf{K_K^h} = (\mathbf{\Pi}_*^\nabla)^T \tilde{\mathbf{G}} \mathbf{\Pi}_*^\nabla + (\mathbf{I} - \mathbf{\Pi}^\nabla)^T (\mathbf{I} - \mathbf{\Pi}^\nabla)$$

- Exact consistency term, approximation for stability term
- Optimal convergence rates in H^1, L^2
- General bilinear forms possible
- 3d: similar, but more technical

Thank you.

Veiga, L. Beirao da, F. Brezzi, A. Cangiani, et al. (2013). "Basic Principles of Virtual Element Methods". In: *Mathematical Models and Methods in Applied Sciences* 23.01, pp. 199–214. DOI: 10.1142/S0218202512500492.

Veiga, L. Beirao da, F. Brezzi, L. D. Marini, et al. (2014). "The Hitchhiker's Guide to the Virtual Element Method". In: *Mathematical Models and Methods in Applied Sciences* 24.08, pp. 1541–1573.