Basic Principles of Virtual Element Methods

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Continuous Problem

Consider simple Laplace problem:

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \Gamma = \partial\Omega$$

Find $u \in V := H_0^1(\Omega)$, such that

$$a(u, v) = (f, v) \quad \forall v \in V,$$

with $a(u, v) = (\nabla u, \nabla v)$.

✓ Lax-Milgram: Existence & Uniqueness

Discrete Problem: Abstract Framework

- $\{\mathcal{T}_h\}_h$ ... decomposition of Ω into elements K
- $\mathcal{E}_h$ ... set of edges e of $\mathcal{T}_h$
- h ... maximum of diameters of elements in $\mathcal{T}_h$

Assumption: A0.1

$\mathcal{T}_h$ is made of a finite number of **simple polygons**.

More precise:

Open simply connected sets with non-intersecting boundaries made of a finite number of straight line segments.

Discrete Problem: Definitions

- $a(u, v) = \sum_{K \in \mathcal{T}_h} a^K(u, v) \qquad \forall u, v \in V$
- $|v|_1^2 = \sum_K |v|_{1,K}^2 \qquad \forall v \in V$
- $|v|_{h,1}^2 := \sum_K |\nabla v|_{0,K}^2 \qquad \forall v \in H^1(\mathcal{T}_h) := \prod_K H^1(K)$

Discrete Problem: Assumptions

Assumption: A1

For each h, we have:

- $V_h \subset V$
- $a_h : V_h \times V_h \to \mathbb{R}$, symmetric bilinear form
- $a_h(u_h, v_h) = \sum_K a_h^K(u_h, v_h) \quad \forall u_h, v_h \in V_h$

  with a_h^K a bilinear form on $V_{h|K} \times V_{h|K}$
- $f_h \in V_h'$

Discrete Problem: More Assumptions

Assumption: A2

There exists $k \geq 1$ such that for all h and $K \in \mathcal{T}_h$

- $\mathbb{P}_k(K) \subset V_{h|K}$
- k-consistency:

  $$a_h^K(p, v_h) = a^K(p, v_h) \quad \forall p \in \mathbb{P}_k(K), v_h \in V_{h|K}$$

- Stability:

  $$\alpha_* a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* a^K(v_h, v_h) \quad \forall v_h \in V_{h|K}$$

Implies continuity of a_h with constant α^*.

Discrete Problem: Error Estimate

Theorem

Under Assumptions **A1-A2**, the discrete problem: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_h,$$

has a unique solution.

For every approximation $u_I \in V_h$ and $u_\pi \in \mathbb{P}_k$ piecewise, we have

$$|u - u_h|_1 \leq C \left(|u - u_I|_1 + |u - u_\pi|_{h,1} + \mathcal{F}_h \right),$$

with $\mathcal{F}_h := \|f - f_h\|_{V_h'}$.

Discretization: Discrete Spaces

- K ... simple polygon with n edges
- x_K ... barycenter of K
- h_K ... diameter of K

$$\mathbb{B}_k(\partial K) := \{v \in C^0(\partial K) : v_{|e} \in \mathbb{P}_k(e) \quad \forall e \subset \partial K\}$$
$$dim = n + n(k-1) = nk$$

$$V^{K,k} := \{v \in H^1(K) : v_{|\partial K} \in \mathbb{B}_k(\partial K), \Delta v_{|K} \in \mathbb{P}_{k-2}(K)\}$$
$$dim = nk + k(k-1)/2 =: N^K$$

Discretization: Degrees of Freedom

For $V^{K,k}$ we can chose the following degrees of freedom:

- $\mathcal{V}^{K,k}$: v_h at **vertices**
- $\mathcal{E}^{K,k}$: v_h at $k-1$ uniformly spaced points on **edges** e
- $\mathcal{P}^{K,k}$: **moments** $\frac{1}{|K|}\int_K m(x)v_h(x)dx \quad \forall m \in \mathcal{M}_{k-2}(K)$

$$\mathcal{M}^{k-2} := \{(\tfrac{x-x_K}{h_K})^s, |s| \leq k-2\}, \quad dim = (k^2-k)/2$$

Discretization: Degrees of Freedom

Remark

- $\mathcal{V}^{K,k} + \mathcal{E}^{K,k} \Leftrightarrow$ prescribe v_h on ∂K
- $\mathcal{P}^{K,k} \Leftrightarrow$ prescribe $P_{k-2}^{K} v_h$ in K

$P_{k-2}^{K} := L^2(K)$ – projection onto $\mathbb{P}_{k-2}(K)$

Theorem

The degrees of freedom are **unisolvent** for $V^{K,k}$.

Discretization: Unisolvence

Theorem

The degrees of freedom are **unisolvent** for $V^{K,k}$.

1. Observe:

$$\text{unisolvence} \Leftrightarrow v_h = 0 \text{ on } \partial K, P_{k-2}^K = 0 \text{ in } K \implies v_h = 0 \text{ in } K$$

2. Show $P_{k-2}^K = 0 \implies \Delta v_h = 0$ in K
   2.1 auxiliary problem, define R
   2.2 show: R is an isomorphism
   2.3 $\Delta v_h = 0$
3. $\implies$ unisolvence

Discretization: Construction of the Discrete Space

$$V_h := \{v \in V : v_{|\partial K} \in \mathbb{B}_k(\partial K) \text{ and } \Delta v_{|K} \in \mathbb{P}_{k-2}(K) \quad \forall K \in \mathcal{T}_h\}$$

$$dim = N_V + N_E(k-1) + N_P \frac{k(k-1)}{2} =: N^{tot}$$

- N_V, N_E, N_P ... number of **internal** vertices, edges and elements

For V_h we can chose the following degrees of freedom:

- $\mathcal{V}$: v_h at **internal vertices**
- $\mathcal{E}$: v_h at $k-1$ uniformly spaced points on **internal edges** e
- $\mathcal{P}$: **moments** $\frac{1}{|K|} \int_K q(x) v_h(x) dx \quad \forall q \in \mathcal{M}_{k-2}(K)$

Discretization: Projection Error

Assumption: A0.2

Assume there exists $\gamma > 0$ such that for all h, each $K \in \mathcal{T}_h$ is star-shaped with respect to a ball of radius $\geq \gamma h_K$.

Theorem: Scott-Dupont

A0.2 implies, that there is a constant $C(\gamma, k)$, such that for every $1 \leq s \leq k+1$, $w \in H^s(K)$, there exists $w_\pi \in \mathbb{P}_k(K)$ such that

$$\|w - w_\pi\|_{0,K} + h_K |w - w_\pi|_{1,K} \leq C h_K^s |w|_{s,K}.$$

Discretization: Interpolation Error

Theorem: Brenner-Scott

Assume A0.2. Then there is a constant $C(\gamma, k)$, such that for all $2 \leq s \leq k+1$, h, $K \in \mathcal{T}_h$, $w \in H^s(K)$, there exists $w_I \in V^{K,k}$ such that

$$\|w - w_I\|_{0,K} + h_K |w - w_I|_{1,K} \leq C h_K^s |w|_{s,K}.$$

Discretization: Construction of a_h

- Did not specify a_h so far!
- Only knowledge: has to satisfy A2 (consistency, stability)

For $p \in \mathbb{P}_k(K), v \in V^{K,k}$, we observe:

$$a^K(p, v) = \int_K \nabla p \cdot \nabla v dx = -\int_K \Delta p \; v dx + \int_{\partial K} \frac{\partial p}{\partial n} v ds$$

- $\Delta p \in \mathbb{P}_{k-2}(K)$ and $\frac{\partial p}{\partial n} \in \mathbb{P}_{k-1}(e)$
- Can compute without knowing v in the interior of K!

  (via moments and edge values)

Discretization: Construction of a_h

- $\overline{\varphi} := \frac{1}{n}\sum_{i=1}^{n} \varphi(V_i)$
- $\Pi_K^K : V^{K,k} \to \mathbb{P}_k(K) \subset V^{K,k}$

$$\begin{cases} a^K(\Pi_k^K v, q) = a^K(v, q) \quad \forall q \in \mathbb{P}_k(K) \\ \overline{\Pi_k^K v} = \overline{v} \end{cases}$$

- We have $\Pi_k^K q = q \quad \forall q \in \mathbb{P}_k(K)$
- Choice: $a_h^K(u, v) := a^K(\Pi_h^K u, \Pi_k^K v)$
  - ✓ k-consistency
  - ✗ stability

Discretization: Construction of a_h

- Chose $S^K(u,v)$ symmetric, positiv, bilinear form such that

$$c_0 a^K(v,v) \leq S^K(v,v) \leq c_1 a^K(v,v) \quad \forall v \in V^{K,k} \text{ with } \Pi_k^K v = 0$$

- Define

$$a_h^K(u,v) := a^K(\Pi_k^K u, \Pi_k^K v) + S^K(u - \Pi_k^K u, v - \Pi_k^K v)$$

Theorem: Discrete Bilinear-Form

✓ k-consistency

✓ stability

Discretization: Choice of S_K

Assumption: A0.3

Assume that there is a $\gamma > 0$ such that for all h and each $K \in \mathcal{T}_h$, the distance between any two vertices of K is $\geq \gamma h_K$.

- In general: S^K depends on problem
- S^K must scale like a^K on the kernel of Π_k^K
- $a^K(\varphi_i, \varphi_j) \simeq 1$

$$S^K(\varphi_i - \Pi_k^K \varphi_i, \varphi_j - \Pi_k^K \varphi_j) := \sum_{r=1}^{N^K} \mathcal{X}_r(\varphi_i - \Pi_k^K \varphi_i)\, \mathcal{X}_r(\varphi_j - \Pi_k^K \varphi_j)$$

Discretization: Right-Hand Side

- Define right-hand side

$$f_h := P^K_{k-2} f \quad \text{on each } K \in \mathcal{T}_h$$

Then:

$$\langle f_h, v_h \rangle = \sum_K \int_K f_h v_h dx = \sum_K \int_K (P^K_{k-2} f) v_h dx = \sum_K \int_K f (P^K_{k-2} v_h) dx$$

- only need internal moments
- Furthermore:

$$\mathcal{F}_h \leq C h^k \left(\sum_K |f|^2_{k-1,K} \right)^{\frac{1}{2}}$$

Conclusions

- Optimal order also for L^2 possible
- Complicated geometries
- Higher-order continuity
- Replace Δ in $V^{K,k}$ by second-order elliptic operator
- Even further: just require that
  - $dim V^{K,k} = N^K$
  - contains polynomials $\leq k$ on e
  - contains $\mathbb{P}_k$
  - unisolvent

. . . to be continued . . .

this December: "**The Hitchhikers Guide to VEM**"

Thank you.

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