CISM COURSE COMPUTATIONAL ACOUSTICS

Solvers

Part 4: Multigrid I

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Outline

Outline

A first idea

Idea: Analyze the damped Jacobi method in more detail

Simplification: 1d-Poisson problem:

- $\Omega = (0,1)$, V_0 continuous and piecwise linear functions
- Find $u \in V_0 : \int_0^1 u'(x)v'(x)\mathrm{d}x = \int_0^1 f(x)v(x)\mathrm{d}x \; \forall v \in V_0$

Linear system:

$$\mathbf{K}\underline{u} = \underline{f},$$

with

$$\mathbf{K} = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \text{ and } \underline{f} = \left[\int_0^1 f(x)N_i(x)\mathrm{d}x\right]_{i=1}^{n_h}.$$

A first idea

Damped Jacobi method

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\, \mathbf{D}^{-1} \left[\underline{f} - \mathbf{K}\underline{u}^{(k)} \right] \quad \text{for } k = 0, 1, \ldots$$

- Use $\underline{f} = 0$ and $\underline{u}^{(0)} = [\mathrm{rand}(0,1)]_{j=1}^{n_h}$.
- Apply Jacobi method for $\alpha = 1$:

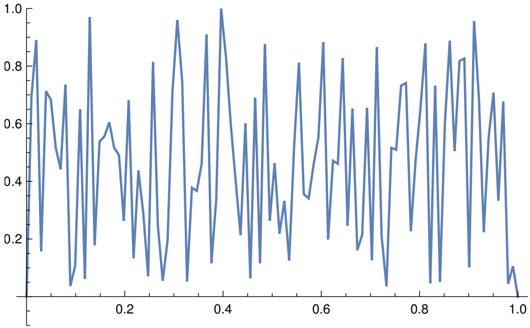

Figure: $k = 0$

A first idea

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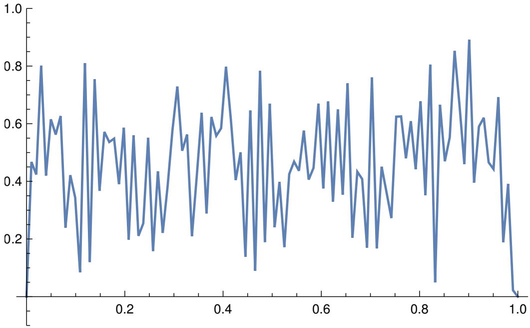

Figure: $k = 1$

A first idea

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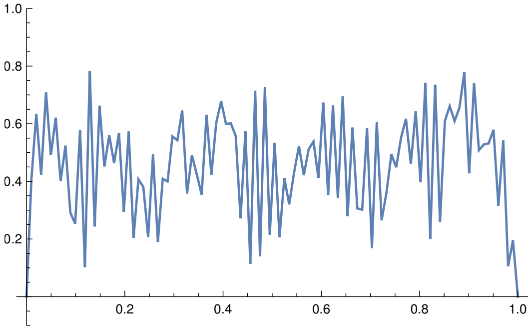

Figure: $k = 2$

A first idea

Damped Jacobi method

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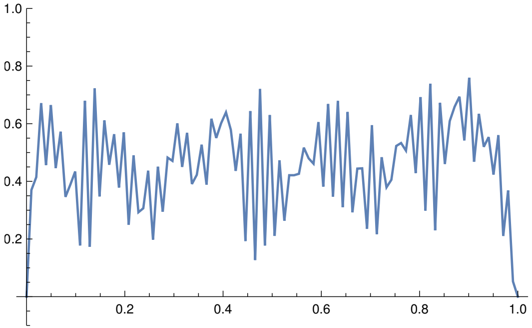

Figure: $k = 3$

A first idea

Damped Jacobi method

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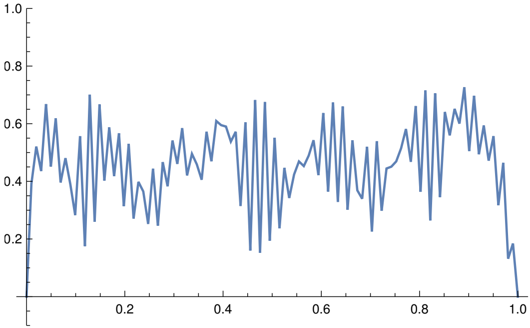

Figure: $k = 4$

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Damped Jacobi method

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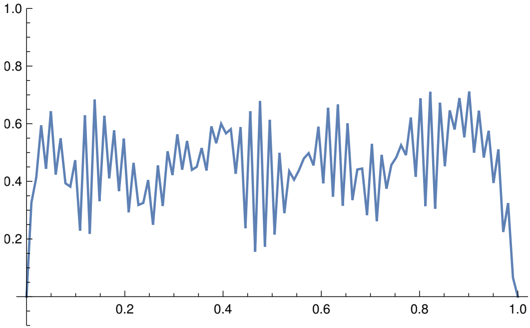

Figure: $k = 5$

A first idea

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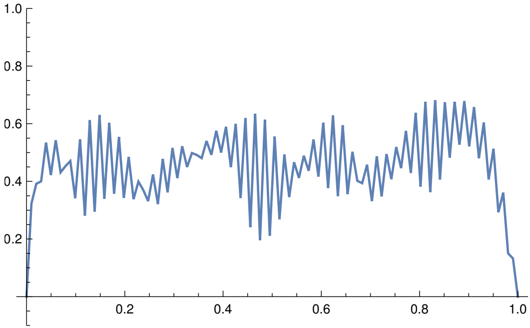

Figure: $k = 10$

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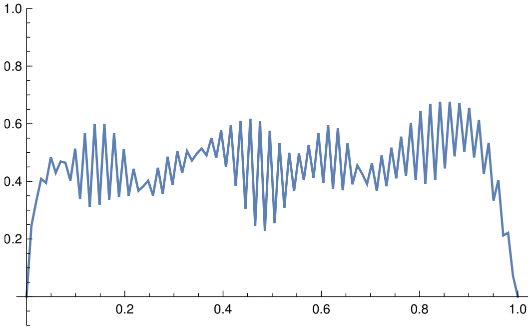

Figure: $k = 15$

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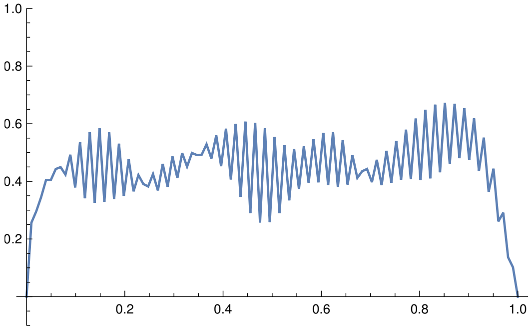

Figure: $k = 20$

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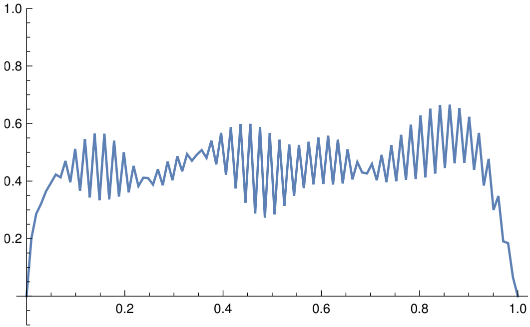

Figure: $k = 25$

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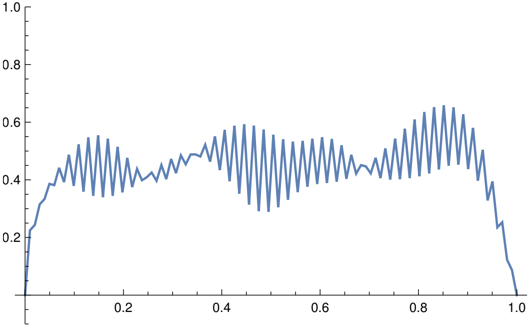

Figure: $k = 30$

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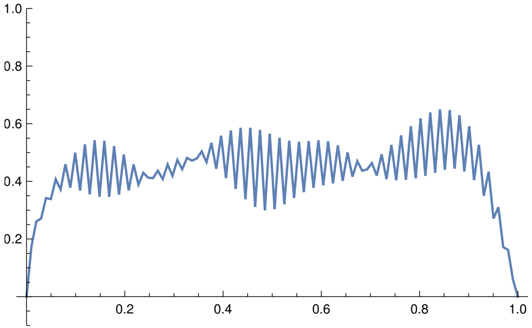

Figure: $k = 35$

A first idea

Damped Jacobi method

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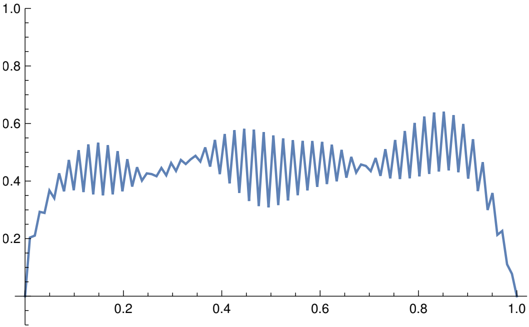

Figure: $k = 40$

A first idea

Damped Jacobi method

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\, \mathbf{D}^{-1} \left[\underline{f} - \mathbf{K} \underline{u}^{(k)} \right] \quad \text{for } k = 0, 1, \ldots$$

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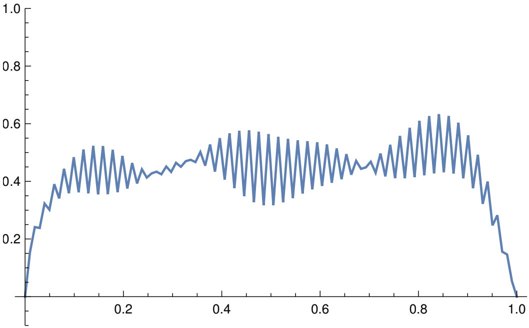

Figure: $k = 45$

Damped Jacobi method

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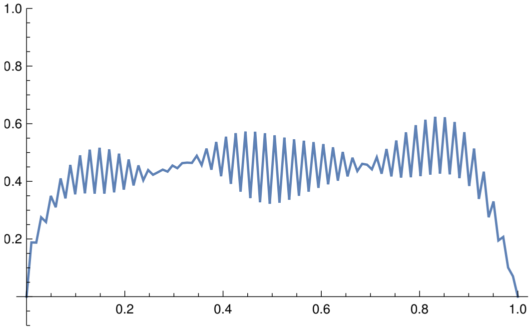

Figure: $k = 50$

A first idea

Damped Jacobi method

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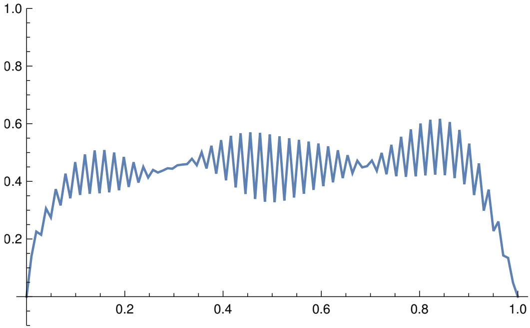

Figure: $k = 55$

A first idea

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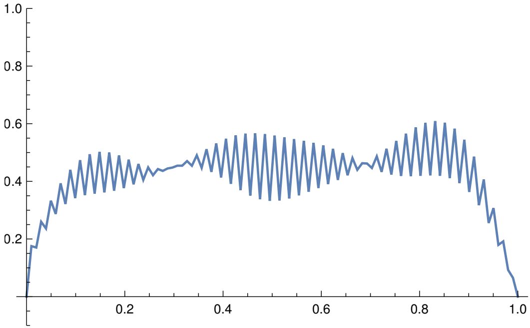

Figure: $k = 60$

A first idea

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- Use $\underline{f} = 0$ and $\underline{u}^{(0)} = [\mathrm{rand}(0,1)]_{j=1}^{n_h}$.
- Apply Jacobi method for $\alpha = \frac{2}{3}$:

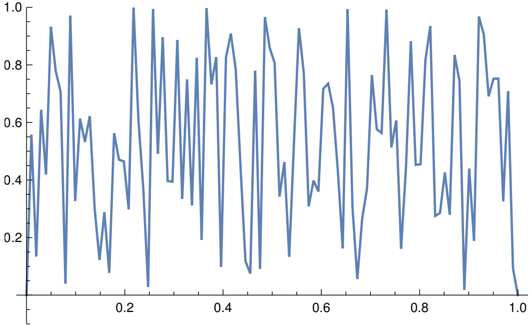

Figure: $k = 0$

A first idea

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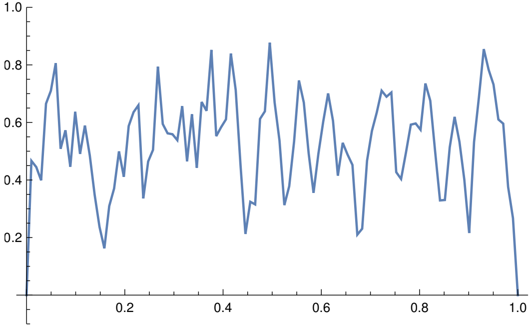

Figure: $k = 1$

A first idea

Damped Jacobi method

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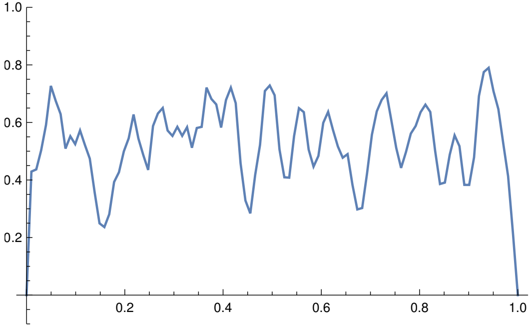

Figure: $k = 2$

A first idea

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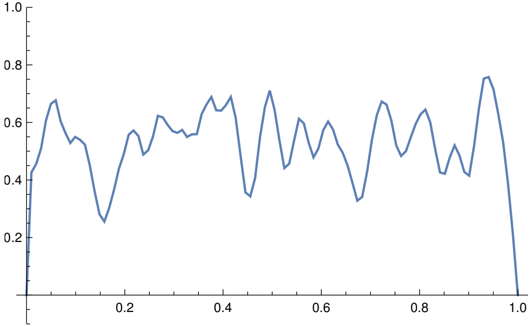

Figure: $k = 3$

A first idea

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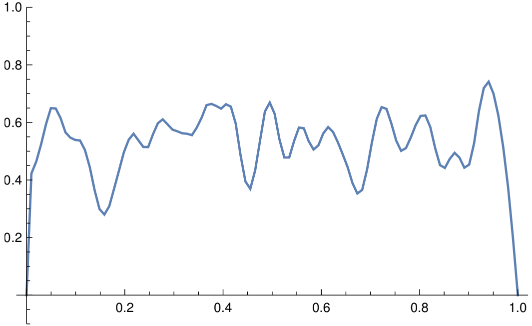

Figure: $k = 4$

A first idea

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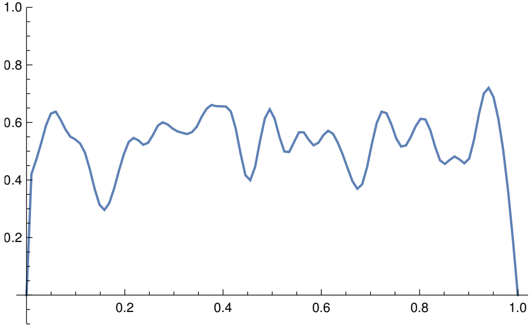

Figure: $k = 5$

A first idea

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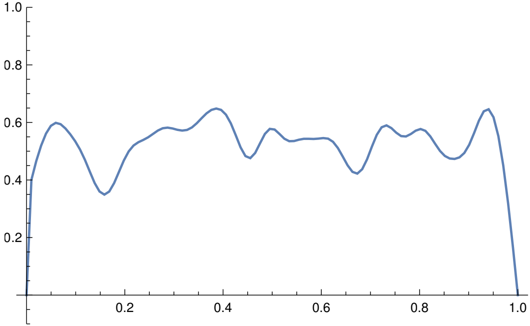

Figure: $k = 10$

A first idea

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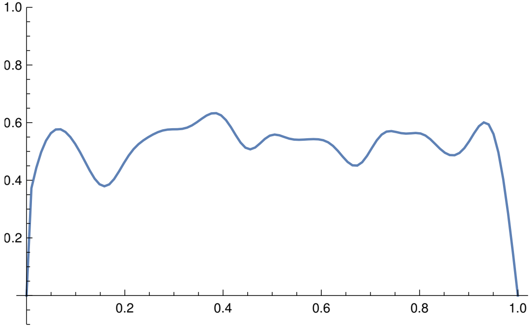

Figure: $k = 15$

A first idea

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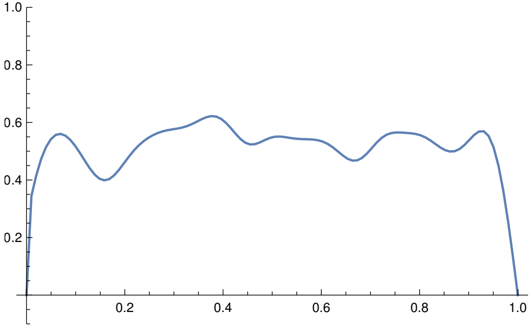

Figure: $k = 20$

A first idea

Damped Jacobi method

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\, \mathbf{D}^{-1} \left[\underline{f} - \mathbf{K}\underline{u}^{(k)} \right] \quad \text{for } k = 0, 1, \ldots$$

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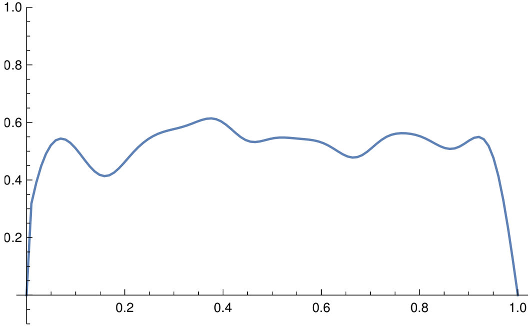

Figure: $k = 25$

A first idea

Damped Jacobi method

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\, \mathbf{D}^{-1} \left[\underline{f} - \mathbf{K}\underline{u}^{(k)} \right] \quad \text{for } k = 0, 1, \ldots$$

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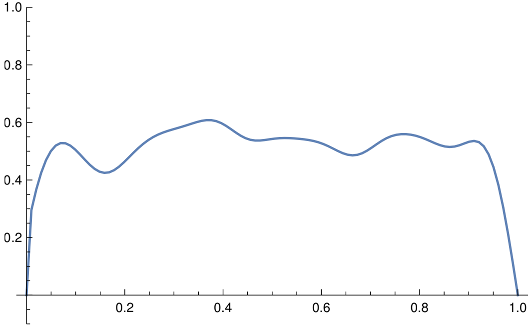

Figure: $k = 30$

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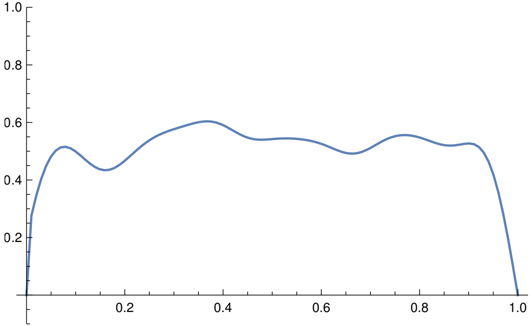

Figure: $k = 35$

A first idea

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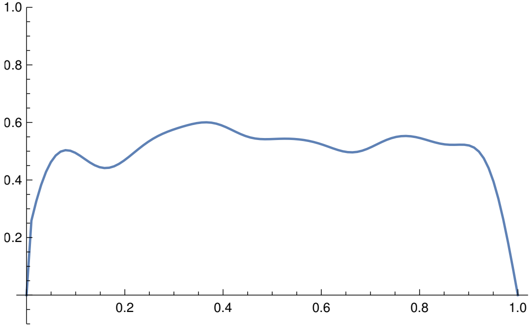

Figure: $k = 40$

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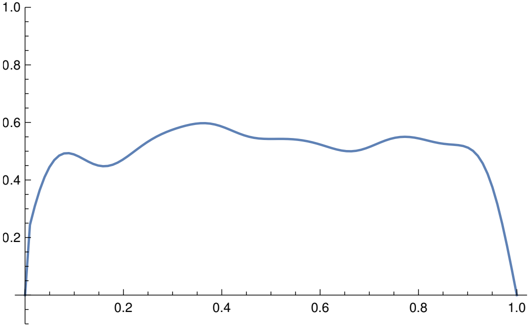

Figure: $k = 45$

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- Use $\underline{f} = 0$ and $\underline{u}^{(0)} = [\mathrm{rand}(0,1)]_{j=1}^{n_h}$.
- Apply Jacobi method for $\alpha = \frac{2}{3}$:

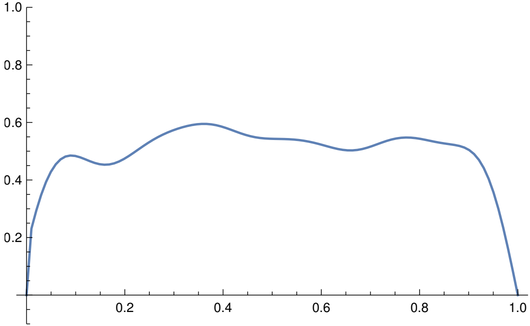


Figure: $k = 50$

A first idea

Damped Jacobi method

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\, \mathbf{D}^{-1} \left[\underline{f} - \mathbf{K}\underline{u}^{(k)} \right] \quad \text{for } k = 0, 1, \ldots$$

- Use $\underline{f} = 0$ and $\underline{u}^{(0)} = [\text{rand}(0,1)]_{j=1}^{n_h}$.
- Apply Jacobi method for $\alpha = \frac{2}{3}$:

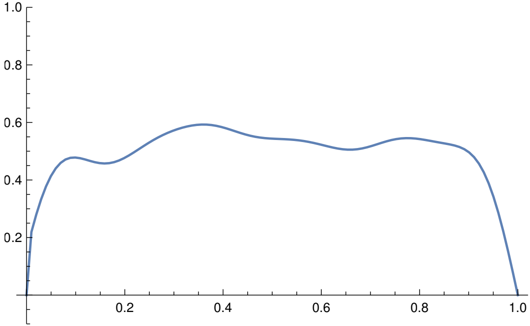


Figure: $k = 55$

A first idea

Damped Jacobi method

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\, \mathbf{D}^{-1} \left[\underline{f} - \mathbf{K}\underline{u}^{(k)} \right] \quad \text{for } k = 0, 1, \ldots$$

- Use $\underline{f} = 0$ and $\underline{u}^{(0)} = [\mathrm{rand}(0,1)]_{j=1}^{n_h}$.
- Apply Jacobi method for $\alpha = \frac{2}{3}$:

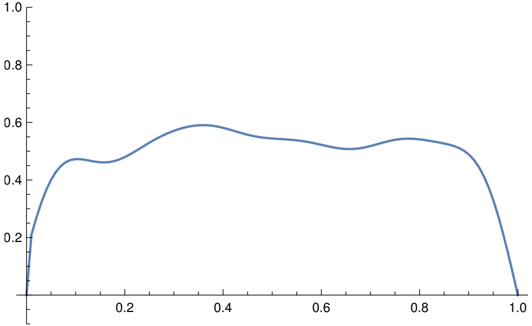


Figure: $k = 60$

A first idea

Observations:

- In all cases the error is converging very slowly
- For $\alpha = \frac{2}{3}$ the error is getting smoother

Explanation: Fourier expansion: $\mathbf{K}\,\underline{\phi}_i = \lambda_i \underline{\phi}_i$ with

$$\lambda_i = \frac{4}{h}\sin^2\left(\frac{i\,\pi}{2n}\right) \quad \text{and} \quad \underline{\phi}_i = \left[\sqrt{2n}\,\sin(ik\pi h)\right]_{k=1}^{n_h}$$

$$\rightarrow \quad \underline{e}^{(0)} := \underline{u}^{(0)} - \underline{u} = \sum_{i=1}^{n_h} \alpha_i \underline{\phi}_i.$$

Error propagation:

$$\underline{e}^{(k+1)} = \mathbf{S}\,\underline{e}^{(k)} = \left[I - \alpha \mathbf{D}^{-1}\,\mathbf{K}\right]\underline{e}^{(k)} = \left[I - \alpha \mathbf{D}^{-1}\,\mathbf{K}\right]^k \sum_{i=1}^{n_h} \alpha_i \underline{\phi}_i$$

$$= \sum_{i=1}^{n_h} \alpha_i \left[1 - \alpha \frac{h}{2}\lambda_i\right]^k \underline{\phi}_i = \sum_{i=1}^{n_h} \alpha_i \left[1 - 2\alpha \sin^2\left(\frac{i\,\pi}{2n}\right)\right]^k \underline{\phi}_i.$$

A frist idea

Estimate:

$$\left|1 - 2\alpha \sin^2\left(\frac{i\,\pi}{2n}\right)\right| \quad \text{for } i = 1, \ldots, n-1$$

- $i = \frac{n}{2}, \ldots, n-1$:

$$\left|1 - 2\alpha \sin^2\left(\frac{i\,\pi}{2n}\right)\right| \leq \max\left\{|1-\alpha|, |1-2\alpha|\right\} = \frac{1}{3} \quad \text{for } \alpha^* = \frac{2}{3}.$$

- $i = 1, \ldots, \frac{n}{2}$:

$$\begin{aligned}\left|1 - 2\alpha \sin^2\left(\frac{i\,\pi}{2n}\right)\right| &\leq \max\left\{\left|1 - 2\alpha \sin^2\left(\frac{\pi}{2n}\right)\right|, |1-\alpha|\right\} \\ &= \mathcal{O}(1 - \alpha\frac{\pi^2}{2}h^2) \approx 1.\end{aligned}$$

- $\rightarrow$ fast reduction of the high oscillating error components
- $\rightarrow$ almost no reduction of the smooth part of the error

Outline

Two-grid cycle

Idea: Damped Jacob method + subspace correction step:

- The damped Jacobi method leads to a “smooth” error
- $\rightarrow$ the correction has to be smooth
- A smooth correction can be good approximated on a coarser grid

---

Algorithm Two-grid cycle

Require: Approximation $\underline{u}^{(k)}$

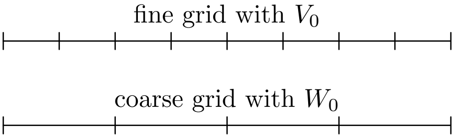
1: Apply **smoothing** procedure $\rightarrow \underline{u}^{(k+1/3)}$

2: Apply **subspace correction** $\rightarrow \underline{u}^{(k+2/3)}$

3: Apply **smoothing** procedure $\rightarrow \underline{u}^{(k+1)}$

Subspace correction

- Given smoothed approximation $u^{(k)} \in V_0 \leftrightarrow \underline{u}^{(k)} \in \mathbb{R}^{n_h}$
- Cosider subspace comming from a coarser grid: $W_0 \subset V_0$



Subspace correction:

$$\underline{w}^{(k)} \in \mathbb{R}^{n_h} \leftrightarrow w^{(k)} \in W_0 : \; a(w^{(k)}, v) = \ell(v) - a(u^{(k)}, v) \quad \forall v \in W_0,$$

with equivalent system of linear equations

$$\mathbf{K}_C \, \underline{w}_C^{(k)} = \underline{r}_C^{(k)}$$

- Connection $\underline{w}_C^{(k)} \in \mathbb{R}^{n_C} \quad \leftrightarrow \quad \underline{w}^{(k)} \in \mathbb{R}^{n_h}$?
- Connection $\underline{r}_C^{(k)} \in \mathbb{R}^{n_C} \quad \leftrightarrow \quad \underline{r}^{(k)} = \underline{f} - \mathbf{K}\,\underline{u}^{(k)} \in \mathbb{R}^{n_h}$?

Connection $\underline{w}_C^{(k)} \in \mathbb{R}^{n_C} \quad \leftrightarrow \quad \underline{w}^{(k)} \in \mathbb{R}^{n_h}$?

For any $w^{(k)} \in W_0 \subset V_0$

$$w^{(k)} = \sum_{i=1}^{n_C} w_i^C N_i^C \qquad \text{or} \qquad w^{(k)} = \sum_{j=1}^{n_h} w_j N_j.$$

Basis transformation:

$$W_0 \ni N_i^C = \sum_{j=1}^{n_h} P[j,i] N_j, \quad \text{with } P[j,i] \in \mathbb{R} \text{ for } j = 1, \ldots, n_h.$$

$$w^{(k)} = \sum_{i=1}^{n_C} w_i^C N_i^C = \sum_{i=1}^{n_C} w_i^C \left[\sum_{j=1}^{n_h} P[j,i] N_j \right]$$

$$= \sum_{j=1}^{n_h} \left[\sum_{i=1}^{n_C} P[j,i] w_i^C \right] N_j = \sum_{j=1}^{n_h} \left[\mathbf{P} \underline{w}_C^{(k)} \right]_j N_j.$$

Hence we have

$$\underline{w}^{(k)} = \mathbf{P} \underline{w}_C^{(k)}, \quad \text{with } \textbf{prolongation} \text{ matrix } \mathbf{P} \in \mathbb{R}^{n_h \times n_C}.$$

Connection $\underline{r}_C^{(k)} \in \mathbb{R}^{n_C} \quad \leftrightarrow \quad \underline{r}^{(k)} \in \mathbb{R}^{n_h}$?

Consider the coarse grid residual

$$\underline{r}_C^{(k)} \in \mathbb{R}^{n_C} \quad \leftrightarrow \quad \langle R^{(k)}, v\rangle := \ell(v) - a(u^{(k)}, v) \quad \text{for all } v \in W_0.$$

We have

$$\begin{aligned} \underline{r}_C^{(k)}[i] &= \langle R^{(k)}, N_i^C\rangle = \langle R^{(k)}, \sum_{j=1}^{n_h} P[j,i] N_j\rangle \\ &= \sum_{j=1}^{n_h} P[j,i]\langle R^{(k)}, N_j\rangle = \sum_{j=1}^{n_h} P[j,i]\underline{r}^{(k)}[j] = \left[\mathbf{P}^\top \underline{r}^{(k)}\right]_i . \end{aligned}$$

Hence we have

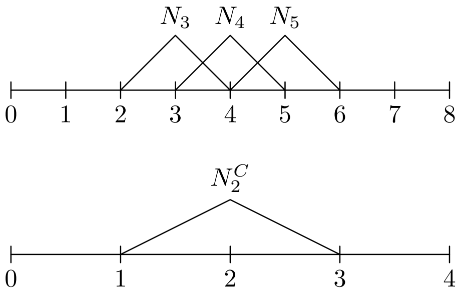
$$\underline{r}_C^{(k)} = \mathbf{P}^\top \underline{r}^{(k)} =: \mathbf{R}\,\underline{r}^{(k)} = \mathbf{R}\left[\underline{f} - \mathbf{K}\,\underline{u}^{(k)}\right]$$

with the **restriction** matrix $\mathbf{R} := \mathbf{P}^\top \in \mathbb{R}^{n_C \times n_h}$.

Grid transfer operators

Basis transformation:

$$W_0 \ni N_i^C = \sum_{j=1}^{n_h} P[j,i] N_j, \quad \text{with } P[j,i] \in \mathbb{R} \text{ for } j = 1, \ldots, n_h.$$



For example:

$$P[3,2] = \frac{1}{2}, \qquad P[4,2] = 1, \qquad P[5,2] = \frac{1}{2}.$$

Grid transfer operators

Prolongation and **restriction** matrices:

$$\mathbf{P} = \begin{pmatrix} 1 & & & & & 0 \\ \frac{1}{2} & & & & & \\ 1 & & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & \\ & 1 & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & \\ & & & \ddots & & \\ & & & & \frac{1}{2} & \frac{1}{2} \\ 0 & & & & & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{R} = \mathbf{P}^\top.$$

- $\mathbf{P}$ and $\mathbf{R}$ are sparse matrices
- $\rightarrow$ Grid transfer is of optimal complexity

Two-grid cycle

This results in the following algorithm:

Algorithm Two-grid cycle

Require: Approximation $\underline{u}^{(k)}$, $\underline{f}$

1: Pre-smoothing: $\underline{u}^{(k)} = S^{\nu}(\underline{u}^{(k)}, \underline{f})$
2: Compute defect: $\underline{d}^{(k)} = \underline{f} - \mathbf{K}\,\underline{u}^{(k)}$
3: Restriction: $\underline{d}_C = \mathbf{R}\,\underline{d}^{(k)}$
4: Solve coarse problem: $\mathbf{K}_C \underline{w}_C = \underline{d}_C$
5: Prolongation: $\underline{w}^{(k)} = \mathbf{P}\,\underline{w}_C$
6: Correction: $\underline{u}^{(k)} = \underline{u}^{(k)} + \underline{w}^{(k)}$
7: Post-smoothing: $\underline{u}^{(k)} = S^{\nu}(\underline{u}^{(k)}, \underline{f})$

- Convergence?
- What to do if coarse problem is still to large?

Two-grid analysis

Two possible ways:

- Fourier analysis (using eigenvalues and eigenvectors of $\mathbf{K}$) $\rightarrow$ additive splitting
- Multiplicative splitting

Need: iteration matrix for the error

Start with smoother: damped Jacobi method:

Consider exact solution $\underline{u} \in \mathbb{R}^{n_h}$ and approximation $\underline{u}^{(k)} \in \mathbb{R}^{n_h}$.

Then we have

$$\begin{aligned} \underline{e}^{(k+1)} := \underline{u}^{(k+1)} - \underline{u} &= \underline{u}^{(k)} + \alpha\, \mathbf{D}^{-1} \left[\underline{f} - \mathbf{K}\, \underline{u}^{(k)} \right] - \underline{u} \\ &= \underline{u}^{(k)} - \underline{u} + \alpha\, \mathbf{D}^{-1}\mathbf{K} \left[\underline{u} - \underline{u}^{(k)} \right] \\ &= \left[I - \alpha\, \mathbf{D}^{-1}\mathbf{K} \right] \underline{e}^{(k)} =: \mathbf{S}\underline{e}^{(k)} = \ldots = \mathbf{S}^k \underline{e}^{(0)}. \end{aligned}$$

Two-grid analysis

Coarse grid correction:

$$\underline{e}_{\mathrm{cor}}^{(k)} := \left(\underline{u}^{(k)} + \underline{w}^{(k)}\right) - \underline{u} = \underline{e}^{(k)} + \underline{w}^{(k)} = \underline{e}^{(k)} + \mathbf{P}\,\underline{w}_C$$

$$= \underline{e}^{(k)} + \mathbf{P}\,\mathbf{K}_C^{-1}\,\underline{d}_C$$

$$= \underline{e}^{(k)} + \mathbf{P}\,\mathbf{K}_C^{-1}\,\mathbf{R}\,\underline{d}^{(k)} = \underline{e}^{(k)} + \mathbf{P}\,\mathbf{K}_C^{-1}\,\mathbf{R}\left[\underline{f} - \mathbf{K}\,\underline{u}^{(k)}\right]$$

$$= \underline{e}^{(k)} - \mathbf{P}\,\mathbf{K}_C^{-1}\,\mathbf{R}\,\mathbf{K}\,\underline{e}^{(k)} = \left[I - \mathbf{P}\,\mathbf{K}_C^{-1}\,\mathbf{R}\,\mathbf{K}\right]\underline{e}^{(k)}$$

$$=: \mathbf{T}\underline{e}^{(k)}.$$

Error of the two-grid cycle:

$$\underline{e}_{\mathrm{tg}}^{(k+1)} = \mathbf{S}^{\nu}\,\mathbf{T}\,\mathbf{S}^{\nu}\,\underline{e}_{\mathrm{tg}}^{(k)} = \mathbf{S}^{\nu}\left[I - \mathbf{P}\,\mathbf{K}_C^{-1}\,\mathbf{R}\,\mathbf{K}\right]\mathbf{S}^{\nu}\,\underline{e}_{\mathrm{tg}}^{(k)} =: \mathbf{M}\,\underline{e}_{\mathrm{tg}}^{(k)}.$$

Estimate:

$$||\underline{e}_{\mathrm{tg}}^{(k+1)}|| \leq ||\mathbf{M}||\;||\underline{e}_{\mathrm{tg}}^{(k)}|| \leq ||\mathbf{M}||^k\;||\underline{e}_{\mathrm{tg}}^{(0)}||.$$

Two-grid analysis

First attempt:

$$||\mathbf{M}|| = ||\mathbf{S}^\nu\,\mathbf{T}\,\mathbf{S}^\nu|| \le ||\mathbf{T}||\;||\mathbf{S}||^{2\nu}.$$

We know

$$||\mathbf{S}||^\nu = [1 - \mathcal{O}(h^\alpha)]^\nu \to 0 \quad \text{for } \nu \to \infty.$$

But:

$$||\mathbf{T}|| = \sup_{0\neq\underline{v}\in\mathbb{R}^{n_h}} \frac{||\mathbf{T}\,\underline{v}||}{||\underline{v}||} = \sup_{0\neq\underline{v}\in\mathbb{R}^{n_h}} \frac{||\left[I - \mathbf{P}\,\mathbf{K}_C^{-1}\,\mathbf{R}\,\mathbf{K}\right]\underline{v}||}{||\underline{v}||}$$

$$\ge \sup_{\substack{0\neq\underline{v}\in\mathbb{R}^{n_h}\\ \mathbf{K}\,\underline{v}\in\ker(\mathbf{R})}} \frac{||\left[I - \mathbf{P}\,\mathbf{K}_C^{-1}\,\mathbf{R}\,\mathbf{K}\right]\underline{v}||}{||\underline{v}||} = 1.$$

Overestimation of

$$||\mathbf{M}||?$$

Two-grid analysis

Better splitting:

$$||\mathbf{T}\,\mathbf{S}^\nu|| = ||\mathbf{T}\,\mathbf{K}^{-1}\,\mathbf{K}\,\mathbf{S}^\nu|| \leq ||\mathbf{T}\,\mathbf{K}^{-1}||\;||\mathbf{K}\,\mathbf{S}^\nu||.$$

- Approximation property

$$||\mathbf{T}\,\mathbf{K}^{-1}|| \leq c\,h^\delta$$

- Smoothing property

$$||\mathbf{K}\,\mathbf{S}^\nu|| \leq \eta(\nu)h^{-\delta} \quad \text{with } \eta(\nu) \to 0 \quad \text{as } \nu \to \infty.$$

Then we have convergence

$$||M|| \leq ||\mathbf{T}\,\mathbf{S}^\nu|| \leq c\,\eta(\nu) < 1,$$

for $\nu \in \mathbb{N}$ large enough.

Two-grid analysis

Assumptions:

- d-dimensional Poisson problem
- Some regularity assumptions ($\rightarrow$ restriction for the domain Ω)

Theorem (Approximation property)

$$||\mathbf{T}\,\mathbf{K}^{-1}|| \leq c_1\, h^{2-d}.$$

Theorem (Smoothing property)

$$||\mathbf{K}\,\mathbf{S}^{\nu}|| \leq \frac{c_2}{\nu}\, h^{d-2}.$$

$\rightarrow$ convergence of two-grid cycle for ν large enough!

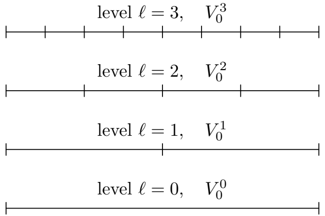
Outline

Multigrid cycle

What to do if coarse problem is still to large?

Idea: Approximate the solution of the coarse grid problem by another two-grid cycle $\rightarrow$ repeat this idea recursively

$\rightarrow$ Need: hierarchy of grids

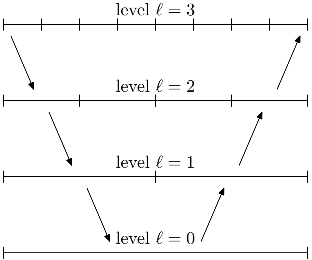


- System matrices $\mathbf{K}_\ell$ on each level $\ell = 0, 1, \ldots, L$.
- Restriction matrix $\mathbf{R}_\ell$ between level ℓ and level $\ell - 1$
- Prolongation matrix $\mathbf{P}_\ell$ between level ℓ and level $\ell - 1$

Solve

$$K_\ell \underline{u}_\ell = \underline{f}_\ell \qquad \text{for } \ell = L.$$

Multigrid cycle



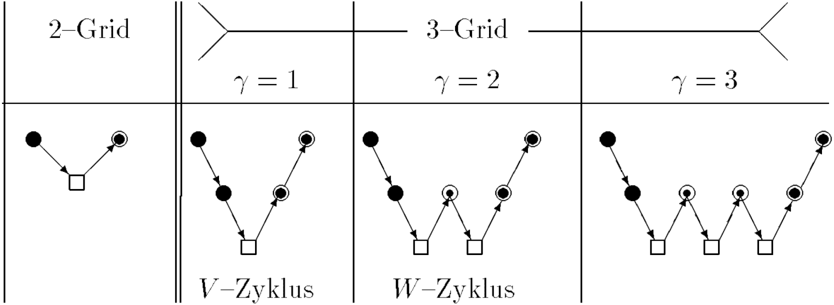
Algorithm MGCycle

Require: $\underline{u}_\ell$, $\underline{f}_\ell$

if $\ell = 0$ **then**
    Coarse grid solver: $\underline{u}_\ell = \mathbf{K}_\ell^{-1} \underline{f}_\ell$
else
    Pre-smoothing: $\underline{u}_\ell = S_\ell(\underline{u}_\ell, \underline{f}_\ell)$
    Compute defect: $\underline{d}_\ell = \underline{f}_\ell - \mathbf{K}_\ell \, \underline{u}_\ell$
    Restriction: $\underline{d}_{\ell-1} = \mathbf{R}_\ell \, \underline{d}_\ell$
    Initialize: $\underline{w}_{\ell-1} = 0$
    for $i = 1, \ldots, \gamma$ **do**
        MGCycle($\underline{w}_{\ell-1}$,$\underline{d}_{\ell-1}$)
    end for
    Prolongation: $\underline{w}_\ell = \mathbf{P}_\ell \, \underline{w}_{\ell-1}$
    Correction: $\underline{u}_\ell = \underline{u}_\ell + \underline{w}_\ell$
    Post-smoothing: $\underline{u}_\ell = S_\ell(\underline{u}_\ell, \underline{f}_\ell)$
end if

Multigrid cycle

Possible cycles:



- $\gamma = 1$ **V-cycle**: cheapest cycle $\rightarrow$ analysis for general problems difficult
- $\gamma = 2$ **W-cycle**: more expensive $\rightarrow$ analysis easier

Multigrid cycle

Full multigrid cycle (Nested iteration)

Idea: Start with coarsest level $\rightarrow$ use as initial guess for the next finer level:

Algorithm Full multigrid cycle

1: Coarse problem: $\underline{u}_0 = \mathbf{K}_0^{-1}\,\underline{f}_0$
2: **for** $\ell = 1, \ldots, L$ **do**
3: Prolongate: $\underline{u}_\ell = \mathbf{P}_\ell\,\underline{u}_{\ell-1}$
4: Apply multigrid-cycle: $\mathrm{MGCycle}(\underline{u}_\ell, \underline{f}_\ell)$
5: **end for**

- Adaptivity $\rightarrow$ construction of the finer grids
- Non-linear problems $\rightarrow$ good initial guess

Outline

Multigrid - example

- $\Omega = (0,1)$, deocmposed with constant mesh size $h_\ell = 2^{-\ell}$
- Find $u \in V_0 : \int_0^1 u'(x)v'(x)\mathrm{d}x = \int_0^1 f(x)v(x)\mathrm{d}x \ \forall v \in V_0$
- Prec. CG-method, rel. residual error reduction $\varepsilon = 10^{-8}$

		MDS		MG	
level	dof n_h	iter	time [s]	iter	time [s]
3	9	5	-	5	-
4	17	11	-	6	-
5	33	16	-	7	-
6	65	20	-	7	-
7	129	22	-	8	-
8	257	24	-	8	-
9	513	26	-	8	-
10	1 025	26	-	8	-
11	2 049	27	0.0015	8	0.0014
12	4 097	29	0.0029	8	0.0024
13	8 193	29	0.0060	8	0.0049
14	16 385	30	0.0131	8	0.0103
15	32 769	32	0.0315	8	0.0255
16	65 537	33	0.0668	9	0.0558
17	131 073	33	0.1377	9	0.1273
18	262 145	34	0.3147	9	0.2359
19	524 289	34	0.6527	9	0.4715
20	1 048 577	35	1.3391	9	0.9583

Outline

Summary

- Two-grid cycle
  - Coarse grid correction
  - Grid transfer operators
  - Two-grid analysis
- Multigrid cycle
- Numerica experiments

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