CISM COURSE COMPUTATIONAL ACOUSTICS

Solvers

Part 3: Preconditioners

Ulrich Langer and **Martin Neumüller**
Institute of Computational Mathematics
Johannes Kepler University Linz
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JOHANNES KEPLER UNIVERSITY LINZ

Institut für NuMa Numerische Mathematik

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Outline

Outline

1. Basic idea

2. Preconditioned iterative methods

- Preconditioned Richardson method
- Preconditioned CG method

3. Subspace correction methods

- Additive-Schwarz methods
- Multiplicative-Schwarz methods

4. Multilevel diagonal scaling

Summary

Basic idea

- **Weak formulation:** Find $u \in V_0 : a(u,v) = \ell(v)\ \forall v \in V_0$
- **Equivalent linear system:**

$$\mathbf{K}\underline{u} = \underline{f}$$

  with $\mathbf{K} \in \mathbb{R}^{n_h \times n_h}$ symmetric and positive definite
- **Condition number:** $\kappa(\mathbf{K}) \to \infty$ as $h \to 0$
- **Iteration numbers:** $\mathcal{O}(\kappa(\mathbf{K})^\alpha) \to \infty$ as $h \to 0$

Idea: Multiply with regular matrix $\mathbf{C^{-1}} \in \mathbb{R}^{n_h \times n_h}$

$$\mathbf{C}^{-1}\mathbf{K}\underline{u} = \mathbf{C^{-1}}\underline{f},$$

such that

$$\kappa(\mathbf{C}^{-1}\mathbf{K}) \leq c \neq c(h).$$

Basic idea

Preconditioned linear system:

$$\mathbf{C}^{-1}\mathbf{K}\underline{u} = \mathbf{C}^{-\mathbf{1}}\underline{f},$$

Requirements:

- Reduce condition number: $\kappa(\mathbf{C}^{-1}\mathbf{K}) \leq c \neq c(h)$
- Cheap realization of $\mathbf{C}^{-1}$, i.e. with complexity

$$\mathcal{O}(n_h) \quad \text{or} \quad \mathcal{O}(n_h \log(n_h)).$$

Basic idea

Lemma

For $\mathbf{K}, \mathbf{C} \in \mathbb{R}^{n_h \times n_h}$ *symmetric and positive definite let the spectral equivalence inequalities be fulfilled, i.e.*

$$c_1(\mathbf{C}\underline{v}, \underline{v}) \leq (\mathbf{K}\underline{v}, \underline{v}) \leq c_2(\mathbf{C}\underline{v}, \underline{v}) \quad \forall \underline{v} \in \mathbb{R}^{n_h}.$$

Then there holds the estimate

$$\kappa(\mathbf{C}^{-1}\mathbf{K}) \leq \frac{c_2}{c_1}.$$

Preconditioner overview

- Algebraic preconditioners:
  - Incomplete LU-factorization (ILU)
  - Incomplete Cholesky-factorization (IC)
  - **Algebraic multigrid method** (AMG)
  - ...
- Preconditioners using variational backround:
  - Schwarz methods
  - **Multilevel methods** (BPX, MDS, AMLI,...)
  - **Multigrid methods** (GMG, AMG)
  - ...

Outline

Preconditioned Richardson method

Applying the Richardson method to the preconditioned linear system:

$$\mathbf{C}^{-1}\mathbf{K}\underline{u} = \mathbf{C}^{-\mathbf{1}}\underline{f},$$

gives

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha \left[\mathbf{C}^{-\mathbf{1}}\underline{f} - \mathbf{C}^{-1}\mathbf{K}\,\underline{u}^{(k)}\right] = \underline{u}^{(k)} + \alpha\,\mathbf{C}^{-1}\left[\underline{f} - \mathbf{K}\underline{u}^{(k)}\right]$$

the **preconditioned Richardson method**

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\,\mathbf{C}^{-1}\left[\underline{f} - \mathbf{K}\underline{u}^{(k)}\right] \quad \text{for } k = 0, 1, 2, \ldots$$

Preconditioned CG method

Applying the **CG-method** to the preconditioned linear system:

$$\mathbf{C}^{-1}\mathbf{K}\underline{u} = \mathbf{C}^{\mathbf{-1}}\underline{f},$$

gives the

Algorithm Preconditioned CG-method

$\underline{r}^{(0)} := \underline{f} - \mathbf{K}\underline{u}^{(0)}, \quad \underline{v}^{(0)} := \mathbf{C}^{-1}\underline{r}^{(0)}, \quad \underline{p}^{(0)} := \underline{v}^{(0)}$

for $k = 0, 1 \ldots$ **do**

$\quad \underline{w}^{(k)} = \mathbf{K}\underline{p}^{(k)}$

$\quad \alpha_k = \frac{(\underline{r}^{(k)}, \underline{v}^{(k)})}{(\underline{w}^{(k)}, \underline{p}^{(k)})}$

$\quad \underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha_k \underline{p}^{(k)}$

$\quad \underline{r}^{(k+1)} = \underline{r}^{(k)} - \alpha_k \underline{w}^{(k)}$

$\quad \underline{v}^{(k+1)} = \mathbf{C}^{-1}\underline{r}^{(k)}$

$\quad \beta_k = \frac{(\underline{r}^{(k+1)}, \underline{v}^{(k+1)})}{(\underline{r}^{(k)}, \underline{v}^{(k)})}, \quad \underline{p}^{(k+1)} = \underline{v}^{(k+1)} + \beta_k \underline{p}^{(k)}$

end for

Preconditioned CG method

Theorem (prec. CG-method convergence)

For the preconditioned CG-method there holds the estimate

$$||\underline{u} - \underline{u}^{(k)}||_A \leq \frac{2q^k}{1+q^{2k}} ||\underline{u} - \underline{u}^{(0)}||_A \leq 2q^k ||\underline{u} - \underline{u}^{(0)}||_A,$$

with

$$q = \frac{\sqrt{\kappa(\mathbf{C}^{-1}\mathbf{K})} - 1}{\sqrt{\kappa(\mathbf{C}^{-1}\mathbf{K})} + 1}$$

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- Multiplicative-Schwarz methods

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Summary

Subspace correction methods

Use variational backround:

$$\mathbf{K}\underline{u} = \underline{f} \quad \Leftrightarrow \quad u \in V_0 : a(u,v) = \ell(v)\ \forall v \in V_0.$$

First idea: Use coercivity and boundedness of $a(\cdot,\cdot)$:

$$\begin{aligned} c_1(\mathbf{B}\underline{v},\underline{v}) &:= c_1(v,v)_V \\ &= c_1||v||_V^2 \le a(v,v) = (\mathbf{K}\underline{v},\underline{v}) \le c_2||v||_V^2 = c_2(\mathbf{B}\underline{v},\underline{v}) \end{aligned}$$

for all $\underline{v} \in \mathbb{R}^{n_h}$.

- Spectral equivalence estimate fulfilled for $\mathbf{B}$ ✓
- Efficient realization of $\mathbf{B}^{-1}$ not directly possible for spaces like $V = H^1(\Omega)$ ×
- BEM: the Preconditioner $\mathbf{B}^{-1}$ can often be realized by a boundary integral operator $\to$ operators of inverse order

Subspace correction methods

Let $\underline{u}^{(k)} \in \mathbb{R}^{n_h} \leftrightarrow u^{(k)} \in V_0$ be an **approximation** of

$$\mathbf{K}\underline{u} = \underline{f} \quad \leftrightarrow \quad u \in V_0 : a(u,v) = \ell(v)\ \forall v \in V_0.$$

Second idea: Use a subspace $W_0 \subset V_0$ and the variational problem:

$$\underline{w}^{(k)} \in \mathbb{R}^{n_h} \leftrightarrow w^{(k)} \in W_0 :\ a(w^{(k)}, v) = \ell(v) - a(u^{(k)}, v) \quad \forall v \in W_0.$$

- If $W_0 = V_0$, then

$$\underline{u} = \underline{u}^{(k)} + \underline{w}^{(k)} \in \mathbb{R}^{n_h} \quad \leftrightarrow \quad u = u^{(k)} + w^{(k)} \in V_0.$$

- This motivates to define for $W_0 \subset V_0$ and $\alpha > 0$ the correction

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\,\underline{w}^{(k)} \in \mathbb{R}^{n_h} \quad \leftrightarrow \quad u^{(k+1)} = u^{(k)} + \alpha\, w^{(k)} \in V_0.$$

If $W_0 \subset V_0$, then **not** all components of V_0 can be corrected $\times$

Subspace correction methods

Third idea: Use a subspace decomposition. Consider the subspaces $W_{0,s} \subset V_0$ for $s = 1, \dots, P$ with

$$V_0 = \sum_{s=1}^{P} W_{0,s} := \left\{ \sum_{s=1}^{P} w_s : w_s \in W_{0,s} \text{ for } s = 1, \dots, P \right\}.$$

For every subspace $W_{0,s}$ we obtain a subspace correction

$$\underline{w}_s^{(k)} \in \mathbb{R}^{n_h} \leftrightarrow w_s^{(k)} \in W_{0,s} :$$
$$a(w_s^{(k)}, v_s) = \ell(v_s) - a(u^{(k)}, v_s) \quad \forall v_s \in W_{0,s}.$$

How to combine all the corrections?

- Additive
- Multiplicative

Additive-Schwarz methods

- Approximation: $\underline{u}^{(k)} \in \mathbb{R}^{n_h} \leftrightarrow u^{(k)} \in V_0$.
- Subspaces
$$V_0 = \sum_{s=1}^{P} W_{0,s}.$$
- Subspace corrections
$$w_s^{(k)} \in W_{0,s} : a(w_s^{(k)}, v_s) = \ell(v_s) - a(u^{(k)}, v_s) \quad \forall v_s \in W_{0,s}.$$

Define the correction

$$w^{(k)} := \sum_{s=1}^{P} w_s^{(k)} \in V_0 \quad \leftrightarrow \quad \underline{w}^{(k)} := \sum_{s=1}^{P} \underline{w}_s^{(k)} \in \mathbb{R}^{n_h}.$$

Next iterate

$$u^{(k+1)} = u^{(k)} + \alpha\, w^{(k)} \in V_0 \quad \leftrightarrow \quad \underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\, \underline{w}^{(k)} \in \mathbb{R}^{n_h}.$$

Additive-Schwarz methods - example

- Discrete space: $V_0 = span\{N_j\}_{j=1}^{n_h}$
- Consider the subspaces

$$W_{0,s} := span\{N_s\} \quad \text{for } s = 1, \ldots, n_h.$$

Then the additive correction is given by

$$w^{(k)} = \sum_{s=1}^{n_h} w_s^{(k)} = \sum_{s=1}^{n_h} w_s N_s \quad \leftrightarrow \quad \underline{w}^{(k)} = [w_s]_{s=1}^{n_h} \in \mathbb{R}^{n_h}.$$

We further obtain the subspace corrections

$$\begin{aligned} & w_s^{(k)} \in W_{0,s}: && a(w_s^{(k)}, v_s) = \ell(v_s) - a(u^{(k)}, v_s) \quad \forall v_s \in W_{0,s}, \\ \Leftrightarrow \quad & w_s \in \mathbb{R} \quad : && a(N_s, N_s) w_s = \ell(N_s) - a(u^{(k)}, N_s), \\ \Leftrightarrow \quad & w_s \in \mathbb{R} \quad : && K_{ss} w_s = f_s - \left[\mathbf{K}\underline{u}^{(k)}\right]_s = \left[\underline{f} - \mathbf{K}\underline{u}^{(k)}\right]_s . \end{aligned}$$

Additive-Schwarz methods - example

Summerizing we have

$$\underline{w}^{(k)} = [w_s]_{s=1}^{n_h} \in \mathbb{R}^{n_h} \quad \text{with} \quad w_s = K_{ss}^{-1} \left[\underline{f} - \mathbf{K}\underline{u}^{(k)} \right]_s .$$

Hence the correction is given by

$$\underline{w}^{(k)} = \mathbf{D}^{-1} \left[\underline{f} - \mathbf{K}\underline{u}^{(k)} \right] \quad \text{with} \quad \mathbf{D} := diag(\mathbf{K}).$$

The next iterate is then given by

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha \underline{w}^{(k)} = \underline{u}^{(k)} + \alpha \mathbf{D}^{-1} \left[\underline{f} - \mathbf{K}\underline{u}^{(k)} \right] .$$

$\rightarrow$ prec. Richardson method with “preconditioner” $\mathbf{D}^{-1}$
or damped Jacobi method.

$\rightarrow$ in general not an optimal method (see prevoiuse lecture)

Multiplicative-Schwarz methods

- Approximation: $\underline{u}^{(k)} \in \mathbb{R}^{n_h} \leftrightarrow u^{(k)} \in V_0$.
- Subspaces

$$V_0 = \sum_{s=1}^{P} W_{0,s}.$$

Algorithm Multiplicative Schwarz

$u_0^{(k)} := u^{(k)}$
for $s = 1, \ldots, P$ **do**
    $w_s^{(k)} \in W_{0,s} : a(w_s^{(k)}, v_s) = \ell(v_s) - a(u_{s-1}^{(k)}, v_s) \; \forall v_s \in W_{0,s}$
    $u_s^{(k)} = u_{s-1}^{(k)} + w_s^{(k)}$
end for
$u^{(k+1)} = u_P^{(k)}$

$\rightarrow$ ordering of the subspaces $W_{0,s}$ plays a role!

Multiplicative-Schwarz methods - example

- Discrete space: $V_0 = span\{N_j\}_{j=1}^{n_h}$
- Consider the subspaces

$$W_{0,s} := span\{N_s\} \quad \text{for } s = 1, \ldots, n_h.$$

Then the correction is given by

$$\underline{w}^{(k)} = \mathbf{L}^{-1}\left[\underline{f} - \mathbf{K}\underline{u}^{(k)}\right], \quad \mathbf{L} := \text{lower triangular mat. of } \mathbf{K}.$$

The next iterate is then given by

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \mathbf{L}^{-1}\left[\underline{f} - \mathbf{K}\underline{u}^{(k)}\right].$$

$\rightarrow$ Gauß-Seidel method

$\rightarrow$ in general not an optimal method (see prevoiuse lecture)

It is possible to combine additive and multiplicative methods, e.g.

$\rightarrow$ Multigrid methods (see later)

Outline

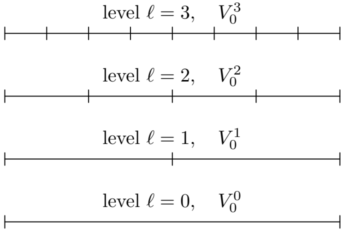
Multilevel diagonal scaling

Simple additive example $\rightarrow$ not efficient preconditioner

Idea: Consider a hierarchy of nested subspaces.

Simplification: 1d-Poisson problem:

- $\Omega = (0,1)$, V_0 continuous and piecwise linear functions
- Find $u \in V_0 : \int_0^1 u'(x)v'(x)\mathrm{d}x = \int_0^1 f(x)v(x)\mathrm{d}x \; \forall v \in V_0$



Nested spaces

$$V_0^0 \subset V_0^1 \subset \ldots \subset V_0^L = V_0.$$

Multilevel diagonal scaling

- Basis functions for each level: $V_0^\ell = span\{N_j^\ell\}_{j=1}^{n_\ell}$.
- For each level we consider the subspaces

$$W_{0,i}^\ell := span\{N_i^\ell\} \quad \text{for } i = 1, \dots, n_\ell, \quad \ell = 0, \dots, L.$$

Subspace decomposition:

$$V_0 = V_L = \sum_{\ell=0}^{L} \sum_{i=1}^{n_\ell} W_{0,i}^\ell.$$

Additive correction:

$$w^{(k)} = \sum_{\ell=1}^{L} \sum_{i=1}^{n_\ell} w_i^\ell N_i^\ell =: \sum_{\ell=1}^{L} w^\ell,$$

with the coefficents from the subspace corrections

$$w_i^\ell \in \mathbb{R} : \quad a(N_i^\ell, N_i^\ell) w_i^\ell = \ell(N_i^\ell) - a(u^{(k)}, N_i^\ell) =: \langle R^\ell, N_i^\ell \rangle =: \left[\underline{r}^\ell \right]_i .$$

Multilevel diagonal scaling

Multi diagonal scaling (MDS) procedure:

- Given approximation

$$\underline{u}^{(k)} \in \mathbb{R}^{n_h} \quad \leftrightarrow \quad u^{(k)} \in V_0.$$

- For each level we apply a diagonal scaling to the residual $\underline{r}^{\ell} := \left[\ell(N_i^{\ell}) - a(u^{(k)}, N_i^{\ell})\right]_{i=1}^{n_\ell}$

$$\underline{w}^{\ell} = \mathbf{D}_{\ell}^{-1} \underline{r}^{\ell} \quad \leftrightarrow \quad w^{\ell} \in V_0^{\ell}.$$

- Sum up all corrections from each level

$$w^{(k)} = \sum_{\ell=0}^{L} w^{\ell} \in V_0 \quad \leftrightarrow \quad \underline{w}^{(k)} \in \mathbb{R}^{n_h}$$

- Compute update

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\, \underline{w}^{(k)}.$$

Multilevel diagonal scaling

- Every computation of one MDS update is linear w.r.t the residual

There exists

$$\mathbf{C}_{\mathrm{MDS}}{}^{-1} : \mathbb{R}^{n_h} \to \mathbb{R}^{n_h},$$

with

$$\underline{w}^{(k)} = \mathbf{C}_{\mathrm{MDS}}{}^{-1}\underline{r} = \mathbf{C}_{\mathrm{MDS}}{}^{-1}\left[\underline{f} - \mathbf{K}\underline{u}^{(k)}\right].$$

This scheme gives the **preconditioned Richardson method**

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha\,\mathbf{C}_{\mathrm{MDS}}{}^{-1}\left[\underline{f} - \mathbf{K}\underline{u}^{(k)}\right] \quad \text{for } k = 0, 1, \ldots$$

- The preconditioner $\mathbf{C}_{\mathrm{MDS}}{}^{-1}$ can be also used in other iterative schmes like the **CG-method.**

Multilevel diagonal scaling

- MDS scheme has optimal complexity $\mathcal{O}(n_h)$
- The MDS scheme is usually implemented by using transfer operators between the different levels $\to$ see later

Theorem

For the MDS preconditioner one can show the spectral equivalence estimates

$$c_1(\mathbf{C}_{\mathrm{MDS}}\underline{v},\underline{v}) \le (\mathbf{K}\underline{v},\underline{v}) \le c_2(\mathbf{C}_{\mathrm{MDS}}\underline{v},\underline{v}) \quad \forall \underline{v} \in \mathbb{R}^{n_h},$$

with constants c_1, c_2 independent of h (only $\log(h)$).

Multilevel diagonal scaling - example

- $\Omega = (0,1)$, deocmposed with constant mesh size $h_\ell = 2^{-\ell}$
- Find $u \in V_0 : \int_0^1 u'(x)v'(x)\mathrm{d}x = \int_0^1 f(x)v(x)\mathrm{d}x \;\forall v \in V_0$
- Prec. CG-method, rel. residual error reduction $\varepsilon = 10^{-8}$

level	dof n_h	iter	time [s]
3	9	5	-
4	17	11	-
5	33	16	-
6	65	20	-
7	129	22	-
8	257	24	-
9	513	26	-
10	1 025	26	-
11	2 049	27	0.0015
12	4 097	29	0.0029
13	8 193	29	0.0060
14	16 385	30	0.0131
15	32 769	32	0.0315
16	65 537	33	0.0668
17	131 073	33	0.1377
18	262 145	34	0.3147
19	524 289	34	0.6527
20	1 048 577	35	1.3391

Outline

Summary

- Basic idea of preconditioning
- Preconditioned iterative methods
- Subspace correction methods
  - Additive
  - Multiplicative
- Multileve diagonal scaling (MDS)

References

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