T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 11 Tuesday, 19 June 2018, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$8 Room: S2 346.

3.6 Variational Crimes

Consider the one-dimensional BVP to find $u \in V_g = V_0 = H_0^1(0,1)$:

$$\int_0^1 \lambda(x)\, u'(x)\, v'(x) dx \; = \; \int_0^1 f(x)\, v(x)\, dx \qquad \forall v \in V_0\,, \tag{3.37}$$

where $f \in L^2(0,\,1)$ and $\lambda \in L^\infty(0,\,1)$. We assume that there exists positive constants $\underline{\lambda}$ and $\overline{\lambda}$ such that $0 < \underline{\lambda} \le \lambda(x) \le \overline{\lambda}$ almost everwhere (a.e.) in (0, 1). Let us consider a finite element discretization with continuous linear finite elements ($\mathcal{F}(\Delta) = \mathcal{P}_1$) on an equidistant grid ($x_i = ih,\; i = \overline{0, n+1}, h = 1/(n+1),\; \delta_r = (x_{r-1}, x_r), r = \overline{1, n+1}$). Now we approximate the bilinear form $a(\cdot,\cdot)$ and the linear form $\langle F, \cdot\rangle$ defined in (3.37) using numerical integration, namely the *midpoint rule*:

$$a_h(u_h,\, v_h) \;=\; \sum_{r=1}^{n+1} h\, \lambda(x_r^\star)\, u_h'(x_r^\star)\, v_h'(x_r^\star) \quad \text{and} \quad \langle F_h,\, v_h\rangle_h \;=\; \sum_{r=1}^{n+1} h\, f(x_r^\star)\, v_h(x_r^\star)\,, \tag{3.38}$$

where $x_r^\star = x_{\delta_r}(\frac{1}{2}) = x_{r-1} + \frac{1}{2}h$. To ensure that these expressions are well-defined we assume (for simplicity) that

$$\lambda,\, f \in W^1_\infty(0,\,1)\,.$$

Let $\widetilde{u}_h \in V_{0h}$ be such that $a_h(\widetilde{u}_h,\, v_h) = \langle F_h,\, v_h\rangle_h$ for all $v_h \in V_{0h}$. We are interested whether the error $\|u - \widetilde{u}_h\|_{H^1(0,1)}$ obeys the same asymptotics with respect to h than if we compute $a(\cdot,\cdot)$ and $\langle F,\,\cdot\rangle$ exactly. This investigation will be done using Strang's first lemma. Throughout this tutorial, we choose $\|\cdot\|_{V_0} := |\cdot|_{H^1(0,1)}$ that is a norm in $V_0 = H_0^1(0,1)$.

57 Show that the bilinear and linear forms above fulfill the standard assumptions (33) (the assumptions of Lax-Milgram) and the additional assumption (34) from the lecture notes. The latter is called *uniform ellipticity* of the discrete bilinear form $a_h(\cdot,\cdot)$, i.e. there exists a positive constant $\mu_3 \ne \mu_3(h)$ such that

$$a_h(v_h,\, v_h) \;\ge\; \mu_3\, \|v_h\|^2_{V_0} \quad \forall v_h \in V_{0h}. \tag{34}$$

Hint: For the uniform ellipticity, use that $\lambda \ge \underline{\lambda} = \text{const} > 0$ and that the midpoint rule is exact for some (which?) polynomials.

We can now apply Strang's first lemma which yields

$$\|u - \widetilde{u}_h\|_{V_0} \;\le\; c \left\{ \inf_{v_h \in V_0} \left[\|u - v_h\|_{V_0} + \sup_{w_h \in V_{0h}} \frac{|a(v_h,\, w_h) - a_h(v_h,\, w_h)|}{\|w_h\|_{V_0}} \right] + \right. \tag{3.39}$$
$$\left. + \sup_{w_h \in V_{0h}} \frac{|\langle F,\, w_h\rangle - \langle F_h,\, w_h\rangle_h|}{\|w_h\|_{V_0}} \right\}$$

58 For $\varphi \in H^1(\delta_r)$, prove that

$$\Big| \int_{\delta_r} \varphi(x)\, dx - h\, \varphi(x_r^*) \Big| \;\le\; c\, h^{3/2}\, |\varphi|_{H^1(\delta_r)}\,,$$

similarly to the exercises in Tutorial 07. Then set $\varphi = f\, w_h$ and show that

$$|\langle F,\, w_h\rangle - \langle F_h,\, w_h\rangle_h| \;\le\; c\, h\, \|f\|_{W^{1,\infty}(0,\,1)}\, \|w_h\|_{V_0}\,.$$

59 For $v_h,\, w_h \in V_{0h}$, prove that

$$|a(v_h,\, w_h) - a_h(v_h,\, w_h)| \;\le\; c\, h\, |\lambda|_{W^1_\infty(0,\,1)}\, \|v_h\|_{V_0}\, \|w_h\|_{V_0}\,.$$

Hint: Treat each element separately and use that $v_h',\, w_h'$ are constant on each element, so that we are left with $| \int_{\delta_r} \lambda(x)\, dx - h\, \lambda(x_r^*)|$. To get an error bound for this term, use Bramble-Hilbert on the reference element.

60 Show that if $u \in H^2(0,\, 1)$ then

$$\|u - \widetilde{u}_h\|_{V_0} \;\le\; c\, h \left\{ |u|_{H^2(0,\,1)} + |\lambda|_{W^1_\infty(0,\,1)}\, \|u\|_{V_0} + \|f\|_{W^1_\infty(0,\,1)} \right\}.$$

Hint: Choose $v_h = u_h$ in (3.39), where $u_h \in V_{0h}$ is the finite element solution in the exact case, i.e.

$$a(u_h,\, v_h) \;=\; \langle F,\, v_h\rangle \qquad \forall v_h \in V_{0h}\,.$$

Show and use that

$$\|u_h\|_{V_0} \le c(\lambda)\, \|u\|_{V_0}\,.$$

3.7 A posteriori error estimates

61 In Section 3.6.2 of our lecture, we derived the residual error estimator for the Dirichlet problem of Poisson's equation. How do we have to modify this estimator such that it works for our model problem CHIP ? Derive the right residual error estimator for the CHIP problem !