T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 08 Tuesday, 29 May 2018, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 346.

DEFINITION 3.3 *A family $\{\tau_h\}_{h\in\Theta}$ of triangulations $\tau_h = \{\delta_r : r \in \mathbb{R}_h\}$ is called regular, if there exists positive and h-independent constants $\underline{c}_1, \overline{c}_1, c_2, c_3 > 0$ such that*

1. $\underline{c}_1 h^d \le |J_{\delta_r}(\xi)| \le \overline{c}_1 h^d$, $\forall \xi \in \overline{\Delta}$,
2. $\|J_{\delta_r}(\xi)\| \le c_2 h$, $\forall \xi \in \overline{\Delta}$,
3. $\|J_{\delta_r}^{-T}(x)\| \le c_3 h^{-1}$, $\forall x \in \overline{\delta}_r$,

and for all $r \in \mathbb{R}_h$ and $h \in \Theta$.

THEOREM 3.4 *Let $a(\cdot,\cdot) : V \times V \to \mathrm{R}^n$ be a bilinear form with $V = H^1(\Omega)$ and $\|\cdot\| = \|\cdot\|_1$, which is symmetric and fulfils the assumptions of Lax Milgram. Moreover, let the triangulation be regular in the sense of Definition 3.3.*
Then the following two statements are valid:

1. *There exists constants $\underline{c}_E, \overline{c}_E > 0$, independent of h such that*
$$\underline{c}_E h^d \le \lambda_{min}(K_h) \le \lambda_{max}(K_h) \le \overline{c}_E h^{d-2},$$
2. $\kappa(K_h) = cond_2(K_h) = \frac{\lambda_{max}(K_h)}{\lambda_{min}(K_h)} \le \frac{\overline{c}_E}{\underline{c}_E} h^{-2}$.

3.3 Properties of the Finite Elements Equations

40 Prove that the inheritance identity

$$(K_h \underline{u}_h, \underline{v}_h) = a(u_h, v_h) \qquad \forall \underline{u}_h, \underline{v}_h \leftrightarrow u_h, v_h \in V_{0h} \tag{3.18}$$

is valid !

41 Show that the eigenvalue estimates in Theorem 3.4 are sharp with respect to the h-order by proving the following statement. There exist positive constants $\underline{c}'_E$ and $\overline{c}'_E$ independent of h satisfying the estimates

$$\lambda_{\min}(K_h) \le \underline{c}'_E h^d \qquad \text{and} \qquad \lambda_{\max}(K_h) \ge \overline{c}'_E h^{d-2}. \tag{3.19}$$

For simplicity, consider the 1D case ($d = 1$):

$$\begin{aligned} -u''(x) &= f(x) \qquad \forall x \in (0,1), \\ u(0) &= u(1) = 0. \end{aligned}$$

42 Show that, for a regular triangulation according to Definition 3.3, there exist h-independent positive constants $\underline{c}_0$ and $\overline{c}_0$ satisfying the inequalities

$$\underline{c}_0 h^d (\underline{v}_h, \underline{v}_h) \;\le\; (M_h \underline{v}_h, \underline{v}_h) \;\le\; \overline{c}_0 h^d (\underline{v}_h, \underline{v}_h) \tag{3.20}$$

for all $\underline{v}_h \in \mathbb{R}^{N_h}$, where M_h denotes the mass matrix defined by the identity

$$(M_h \underline{u}_h, \underline{v}_h) := \int_\Omega u_h(x) v_h(x) dx \qquad \forall \underline{u}_h, \underline{v}_h \leftrightarrow u_h, v_h \in V_{0h}. \tag{3.21}$$

The spectral inequalities (3.21) yield that the mass matrix M_h is well conditioned, i.e., the spectral condition number $\mathrm{cond}_2(M_h)$ can be bounded by the h-independent constant $\overline{c}_0/\underline{c}_0$.

43* Let $\lambda = \lambda_{\max}$ be the maximal eigenvalue of the generalized eigenvalue problem

$$K_h \underline{u}_h = \lambda M_h \underline{u}_h, \tag{3.22}$$

and let $\lambda_r = \lambda_{r,\max}$ be the maximal eigenvalues of generalized eigenvalue problems

$$K_h^{(r)} \underline{u}_h^{(r)} = \lambda_r M_h^{(r)} \underline{u}_h^{(r)}, \tag{3.23}$$

where $K_h^{(r)}$ and $M_h^{(r)}$ denote the (local) element stiffness and mass matrices for element number $r = 1, 2, ..., R_h$, i. e., it holds

$$K_h \;=\; \sum_{r=1}^{R_h} C_r K_h^{(r)} C_r^T \qquad \text{and} \qquad M_h \;=\; \sum_{r=1}^{R_h} C_r M_h^{(r)} C_r^T \,.$$

Show the eigenvalue estimate

$$\lambda \;\le\; \max_{r=1,2,...,R_h} \lambda_r \,. \tag{3.24}$$