T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 04 Tuesday, 17 April 2018, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 346.

17 Let us consider the quadrature rule

$$\int_\Delta u(\xi)d\xi \approx u(\xi^*)|\Delta|,$$

with the unit triangle $\Delta = \{\xi = (\xi_1, \xi_2) \in \mathbf{R}^2 : 0 < \xi_2 < 1 - \xi_1, 0 < \xi_1 < 1\}$ and the integration point $\xi^* = (1/3, 1/3)$. Show that there exists a positive constant $c = const. > 0$ such that

$$|\int_\Delta u(\xi)d\xi - u(\xi^*)|\Delta|\,| \le c\,|u|_{H^2(\Delta)} \quad \forall u \in H^2(\Delta).$$

Hint: In 2D ($d = 2$), $H^2(\Delta)$ is continuously (even compactly) embedded in $C(\overline{\Delta})$, i.e. there exists $c_E = const. > 0 : \|u\|_{C(\overline{\Delta})} := \max_{\xi\in\Delta} |u(\xi)| \le c_E \|u\|_{H^2(\Delta)}$.

18 Let $f \in L_2(\Omega)$ be a given source, and let $g \in H^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))^*$ be a given flux. Show that there exist a unique weak (generalized) solution of the Neumann problem

$$-\Delta u + u = f \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma = \partial\Omega \tag{2.7}$$

satisfying the apriori estimate

$$\|u\|_{H^1(\Omega)} = (\|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)})^{1/2} \le c_1 \|f\|_{L_2(\Omega)} + c_2\,\|g\|_{H^{-1/2}(\Gamma)}.$$

with some positive constant $c_1 = ?$ and $c_2 = ?$.

19 Show that the gradient $q = \nabla u$ of the weak solution u of the Neumann problem (2.7) from Exercise 18 belongs to $H(div)$ and the weak divergence of q is equal to $u - f$, i.e. $\mathrm{div}(q) = u - f$!

20 Let $\Omega_1, \dots, \Omega_m$ be a non-overlapping domain decomposition of Ω, i.e. $\overline{\Omega} = \cup \overline{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, and let $q_i \in H(\mathrm{div}, \Omega_i) \cap C^1(\overline{\Omega}_i)$, $i = 1, 2, \dots, m$, be given functions. Which trace conditions you have to impose on interfaces $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, with $\mathrm{meas}_{d-1}\Gamma_{ij} > 0$, in order to ensure that the piecewise defined function

$$q := \{q|_{\Omega_i} = q_i,\, i = 1, 2, \dots, m\} \in H(\mathrm{div}, \Omega) \text{ and } (\mathrm{div} q)|_{\Omega_i} = \mathrm{div} q_i,$$

for all $i = 1, 2, \dots, m$.

21 Show that, for sufficiently smooth functions, e.g. for $u, v \in H(curl) \cap [C^1(\overline{\Omega})]^3$, the curl-IbyP-formula

$$\int_\Omega \mathrm{curl}(u) \cdot v \, dx = \int_\Omega u \cdot \mathrm{curl}(v) \, dx + \int_\Gamma (u \times n) \cdot v \, ds \qquad (2.8)$$

is valid. **Hint:** Use the classical IbyP-formula for the proof of (2.8) !

22* Let $\Omega_1, \dots, \Omega_m$ be a non-overlapping domain decomposition of Ω, i.e. $\overline{\Omega} = \cup \overline{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, and let $q_i \in H(\mathrm{curl}, \Omega_i) \cap C^1(\overline{\Omega}_i)$, $i = 1, 2, \dots, m$, be given functions. Which trace conditions you have to impose on interfaces $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, with $\mathrm{meas}_{d-1}\Gamma_{ij} > 0$, in order to ensure that the piecewise defined function

$$q := \{q|_{\Omega_i} = q_i,\, i = 1, 2, \dots, m\} \in H(\mathrm{curl}, \Omega) \text{ and } (\mathrm{curl} q)|_{\Omega_i} = \mathrm{curl} q_i,$$

for all $i = 1, 2, \dots, m$.