

T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 03

Tuesday, 10 April 2018, Time: 10¹⁵ – 11⁴⁵, Room: S2 120.

2 Tools from the Theory of Sobolev Spaces

12 Let us consider the function

$$u(x) = \begin{cases} 1, & -1 \leq x \leq 0 \\ -1, & 0 < x \leq 1 \end{cases}.$$

Obviously, $u \in L_p(-1, 1) \subset L_{loc}(-1, 1) \subset D'(-1, 1)$, but $u \notin C[-1, 1]$!
Compute

1. $u' = \partial^1 u \in ?$
2. $u'' = \partial^2 u \in ?$
3. $u''' = \partial^3 u \in ?$

in the distributive sense !

13 Show that

$$\|g\|_{H^{1/2}(\Gamma)} = \inf_{u \in H^1(\Omega): \gamma_0 u = g} \|u\|_{H^1(\Omega)} \quad (2.6)$$

defines a norm in $H^{1/2}(\Gamma) := \gamma_0 H^1(\Omega)$ (check the norm axioms), where $\|u\|_{H^1(\Omega)} := \left(\|u\|_{L_2(\Omega)}^2 + \|\nabla u\|_{L_2(\Omega)}^2 \right)^{1/2}$ denotes the standard norm in $H^1(\Omega)$! The infimum in (2.6) is realized. Characterize the minimizer $u^* \in H^1(\Omega)$ as a unique solution of a variational problem !

14 Show that

$$\|u\|_{W_2^2(\Omega)}^* = \left(\int_{\Gamma_D} |u|^2 ds + \int_{\Gamma_D} |\partial_n u|^2 ds + |u|_{W_2^2(\Omega)}^2 \right)^{1/2}$$

defines a new norm in $W_2^2(\Omega) = H^2(\Omega)$ that is equivalent to the standard norm

$$\|u\|_{W_2^2(\Omega)} = \left(\sum_{|\alpha| \leq 2} \int_{\Omega} |\partial^\alpha u|^2 dx \right)^{1/2} = \left(\int_{\Omega} |u|^2 dx + \int_{\Omega} |\nabla u|^2 dx + |u|_{W_2^2(\Omega)}^2 \right)^{1/2},$$

where $\Gamma_D \subset \Gamma = \partial\Omega$ with $\text{meas}_{d-1}(\Gamma_D) > 0$, $\partial_n u(x) = \frac{\partial u}{\partial n}(x) = (\nabla u(x), n(x)) = \nabla u(x)^T n(x) = \nabla u(x) \bullet n(x)$, and $|u|_{W_2^2(\Omega)} = (\sum_{|\alpha|=2} \int_{\Omega} |\partial^\alpha u|^2 dx)^{1/2}$ denotes the standard semi-norm in $W_2^2(\Omega)$.

- 15** Show that there exists a positive constant $c_0 = \text{const} > 0$ and $c_1 = \text{const} > 0$ such that

$$\int_{\Pi} (u(x))^2 dx \leq c_0 \left(\int_{\Pi} u(x) dx \right)^2 + c_1 \int_{\Pi} |\nabla u(x)|^2 dx \quad \forall u \in H^1(\Pi)$$

with $c_0 = ?$ and $c_1 = ?$, where $\Pi := \{x = (x_1, x_2) \in \mathbf{R}^2 : a_i < x_i < b_i, i = 1, 2\}$.

- Hint: Use the representation

$$u(y_1, y_2) - u(x_1, x_2) = \int_{x_2}^{y_2} \frac{\partial u}{\partial \xi_2}(y_1, \xi_2) d\xi_2 + \int_{x_1}^{y_1} \frac{\partial u}{\partial \xi_1}(\xi_1, x_2) d\xi_1.$$

- 16** Show that the inequalities

$$\inf_{q \in \mathbf{R}} \int_{\Omega} |u(x) - q|^2 dx \leq c^2 \int_{\Omega} |\nabla u(x)|^2 dx \quad \forall u \in W_2^1(\Omega) = H^1(\Omega)$$

and

$$\int_{\Omega} |u(x)|^2 dx \leq \frac{1}{|\Omega|} \left(\int_{\Omega} u(x) dx \right)^2 + c^2 \int_{\Omega} |\nabla u(x)|^2 dx \quad \forall u \in W_2^1(\Omega) = H^1(\Omega)$$

are equivalent !