Analysis of DPG for the Poisson problem

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Overview

Poisson Problem

Ω simply connected, Lipschitz boundary
Ω_h a partition into elements K with Lipschitz boundary

$$-div(\alpha \nabla u) = f \qquad on\ \Omega \tag{1}$$
$$u = 0 \qquad on\ \partial\Omega \tag{2}$$

Assumed for presentation $\alpha = 1$, $f \in L^2$

Ultra Weak Problem

Find $(\underline{\sigma}, u, \hat{u}, \hat{\sigma}_n) \in U = L^2(\Omega) \times L^2(\Omega) \times H_0^{1/2}(\partial\Omega_h) \times H^{-1/2}(\partial\Omega_h)$
such that $\forall(\underline{\tau}, v) \in V = H(div, \Omega_h) \times H^1(\Omega_h)$:

$$(\underline{\sigma}, \underline{\tau})_{\Omega_h} - (u, div\underline{\tau})_{\Omega_h} + \langle \hat{u}, div\underline{\tau} \cdot \underline{n} \rangle_{\partial\Omega_h} - (\underline{\sigma}, \nabla v)_{\Omega_h} + \langle v, \hat{\sigma}_n \rangle_{\partial\Omega_h}$$
$$= (f, v)_{\Omega_h}$$

With element wise scalar products

Appropriate Norms

For V we need broken/element wise norms:

- $\|(v, \underline{q})\|_V^2 = \|v\|_{H^1(\Omega_h)}^2 + \|\underline{q}\|_{H(div,\Omega_h)}^2$
- $\|v\|_{H^1(\Omega_h)}^2 = \sum_{K \in \Omega_h} \|v\|_{H^1(K)}^2$
- $\|\underline{q}\|_{H(div,\Omega_h)}^2 = \sum_{K \in \Omega_h} \|\underline{q}\|_{H(div,K)}^2$

For U we need some trace norms:

- $\|(\underline{\sigma}, u, \hat{u}, \hat{\sigma}_n)\|_U^2 = \|\underline{\sigma}\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\hat{u}\|_{H_0^{1/2}(\partial\Omega_h)}^2 + \|\hat{\sigma}_n\|_{H^{-1/2}(\partial\Omega_h)}^2$
- $\|\hat{u}\|_{H_0^{1/2}(\partial\Omega_h)} = Inf\{\|\omega\|_{H^1(\Omega)} | \omega \in H_0^1(\Omega), trace(\omega) = \hat{u}\}$
- $\|\hat{\sigma}_n\|_{H^{-1/2}(\partial\Omega_h)} = Inf\{\|q\|_{H(div,\Omega)} | q \in H(div, \Omega), trace(q \cdot n) = \hat{\sigma}_n\}$

A Cea-like Lemma

Assume the following conditions for a bilinear form b:

1. $C_1\|v\|_V \leq \sup\limits_{u\in U} \frac{b(u,v)}{\|u\|_U} \qquad \forall v \in V$
2. $\sup\limits_{u\in U} \frac{b(u,v)}{\|u\|_U} \leq C_2\|v\|_V \qquad \forall v \in V$
3. $B : U \to V^* : kern(B) = \{0\}$

Then there exists for a linear functional l a unique solution $u \in U$ to:

$$b(u,v) = l(v) \quad \forall v \in V$$

and a solution $u_h \in U_h \subset U$ to

$$b(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h = T(U_h)$$

With an estimate:

$$\|u - u_h\|_U \leq \frac{C_2}{C_1} \inf_{w_h \in U_h} \|u - w_h\|_U \tag{3}$$

Injectivity

We need to proof:

$$(\forall(\underline{\tau}, v) \in V : b((\underline{\sigma}, u, \hat{u}, \hat{\sigma}_n), (\underline{\tau}, v)) = 0) \implies (\underline{\sigma}, u, \hat{u}, \hat{\sigma}_n) = 0 \qquad (4)$$

Idea for the proof:

1. Compactly supported test function on an element K
2. Find: $\nabla u = div\underline{\sigma} = 0$ in a weak sense
3. $u \in H^1(K), \sigma \in H(div, K)$
4. $u = \hat{u}, \underline{\sigma} \cdot n = \hat{\sigma}_n$ on $\partial\Omega_h$
5. $u \in H_0^1(\Omega), \sigma \in H(div, \Omega)$
6. $u = 0$
7. Integration by parts: $(\underline{\sigma}, \underline{\sigma})_{\Omega_h} = 0 \implies \underline{\sigma} = 0$

Inf-sup Condition

$$C_1\|v\|_V \leq \sup_{(\underline{\sigma},u,\hat{u},\hat{\sigma}_n)\in U} \frac{b((\underline{\sigma},u,\hat{u},\hat{\sigma}_n),(\underline{\tau},v))}{\|(\underline{\sigma},u,\hat{u},\hat{\sigma}_n)\|_U} \qquad \forall(\underline{\tau},v)\in V \tag{5}$$

A Decomposition is the essential step:

$$(\underline{\tau},v) = (\underline{\tau}_0,v_0) + (\underline{\tau}_1,v_1) \tag{6}$$

Into a broken harmonic part and a $H(div,\Omega)\times H_0^1(\Omega)$ part, each handled in a Lemma.

Decomposition

We need $(\tau_1, \sigma_1) \in H(div, \Omega) \times H_0^1$ such that:

$$\underline{\tau}_1 - \nabla v_1 = \underline{\tau} - \nabla v$$
$$div \underline{\tau}_1 = div \underline{\tau}$$

Then we get a decomposition:

$$\underline{\tau} = \underline{\tau}_1 + \underline{\tau}_0$$
$$v = v_1 + v_0$$

Existence and estimates follow on the next slides.

Inhomogeneous Part

$G \in L^2(\Omega)^N$, $F \in L^2(\Omega)$

$$\underline{\tau}_1 - \nabla v_1 = G \qquad on\ \Omega \tag{7}$$

$$div \underline{\tau}_1 = F \qquad on\ \Omega \tag{8}$$

Then it has a unique solution and:

$$\|\underline{\tau}_1\|_{L^2(\Omega)} + \|\nabla v_1\|_{L^2(\Omega)} \leq C(\|G\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}) \tag{9}$$

Proof: Well known problem, Babuška-Aziz

Broken Harmonic Part

Assume $(\underline{\tau}_0, v_0) \in H(div, \Omega_h) \times H^1(\Omega_h)$ fulfills element wise:

$$\underline{\tau}_0 - \nabla v_0 = 0 \tag{10}$$

$$div \underline{\tau}_0 = 0 \tag{11}$$

Then:

$$\|\underline{\tau}_0\|_{L^2(\Omega_h)} = \|\nabla v_0\|_{L^2(\Omega_h)} \leq C(\|[\underline{\tau}_0 \cdot n]\|_{\partial\Omega_h} + \|[v_0 n]\|_{\partial\Omega_h}) \tag{12}$$

Proof: Blackboard

Finishing Inf-sup Proof

Easy manipulations and previous results give:

$$\|\underline{\tau}\|_{L^2(\Omega)} \leq C(\|\underbrace{\underline{\tau} - \nabla v}_{G}\|_{L^2(\Omega)} + \|\underbrace{div\underline{\tau}}_{F}\|_{L^2(\Omega)} + \|[\underline{\tau} \cdot n]\|_{\partial\Omega_h} + \|[vn]\|_{\partial\Omega_h})$$

$$\|\nabla v\|_{L^2(\Omega)} \leq C(\|\underbrace{\underline{\tau} - \nabla v}_{G}\|_{L^2(\Omega)} + \|\underbrace{div\underline{\tau}}_{F}\|_{L^2(\Omega)} + \|[\underline{\tau} \cdot n]\|_{\partial\Omega_h} + \|[vn]\|_{\partial\Omega_h})$$

$$\leq 2C \sup_{(\underline{\sigma}, u, \hat{u}, \hat{\sigma}_n) \in U} \frac{b((\underline{\sigma}, u, \hat{u}, \hat{\sigma}_n), (\underline{\tau}, v))}{\|(\underline{\sigma}, u, \hat{u}, \hat{\sigma}_n)\|_U}$$

$$\|div\underline{\tau}\|_{\Omega_h} is\ part\ of\ the\ right\ side$$

$$\|v\|_{L^2(\Omega)} \leq C(\|\nabla v\|_{\Omega_h} + \|[vn]\|_{\partial\Omega_h})$$

This proofs the inf-sup condition

Boundedness

We need to proof:

$$\|\underline{\tau} - \nabla v\|_{L^2(\Omega)}^2 + \|div\underline{\tau}\|_{L^2(\Omega)}^2 + \|[\underline{\tau} \cdot n]\|_{\partial\Omega_h}^2 + \|[vn]\|_{\partial\Omega_h}^2$$
$$\leq C_2^2(\|\underline{\tau}\|_{H(div,\Omega_h)}^2 + \|v\|_{H^1(\Omega_h)}^2)$$

First two terms are obvious, other two follow from element-wise integration by parts.
All conditions for the Cea-like Lemma have been proven

Approximation for Tetrahedrons

We now have:

$$\|u - u_h\|_U \leq \frac{C_2}{C_1} \inf_{w_h \in U_h} \|u - w_h\|_U$$

$U_h \subset U$ contains piecewise polynomials of some degree,
only $\hat{u}$ is approximated with continuous polynomials
Approximation estimates on a tetrahedral mesh for degree p on each variable:

1. $\inf_{w_h} \|u - w_h\|_{L^2} \leq Ch^s p^{-s} |u|_{H^s(\Omega)} \qquad (s \leq p + 1)$
2. Same for $\underline{\tau}$
3. $\inf_{\hat{z}_h} \|\hat{u} - \hat{z}_h\|_{H_0^{1/2}(\partial\Omega_h)} \leq C \ln(p)^2 h^s p^{-s} |u|_{H^s(\Omega)} \qquad (s \leq p)$
4. $\inf_{\hat{\eta}_{n,h}} \|\hat{\sigma}_n - \hat{\eta}_{n,h}\|_{H^{-1/2}(\partial\Omega_h)} \leq C \ln(p)^2 h^s p^{-s} |\sigma|_{H^s(\Omega)} \qquad (s \leq p + 1)$

Sufficient regularity, degree $p + 1$ for $\hat{u}$, p for the rest
$\implies h^{p+1}$ convergence

DPG-Guideline Summary

Optimal test space norm:

$$\|v\|_{opt,V} := \sup_{u \in U} \frac{b(u,v)}{\|u\|_U} \tag{13}$$

Injective trial to test operator $T : U \to V$:

$$(Tw, v)_V = b(w, v) \quad \forall v \in V \tag{14}$$

For $U_h \subset U$, $V_h = T(U_h)$
Symmetric positive definite system:

$$b(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h \tag{15}$$

Reinterpretation

The inf-sup condition and boundedness can be interpreted as:

$$C_1 \|v\|_V \leq \|v\|_{opt,V} \leq C_2 \|v\|_V \quad \forall v \in V \tag{16}$$

Replacing $\| * \|_V$ by $\| * \|_{opt,V}$ to compute T would be optimal, but is not feasible.
No conclusions about approximating T in this paper,
however the numerical results already use an approximated T operator

Numerical results uniform h-refinement

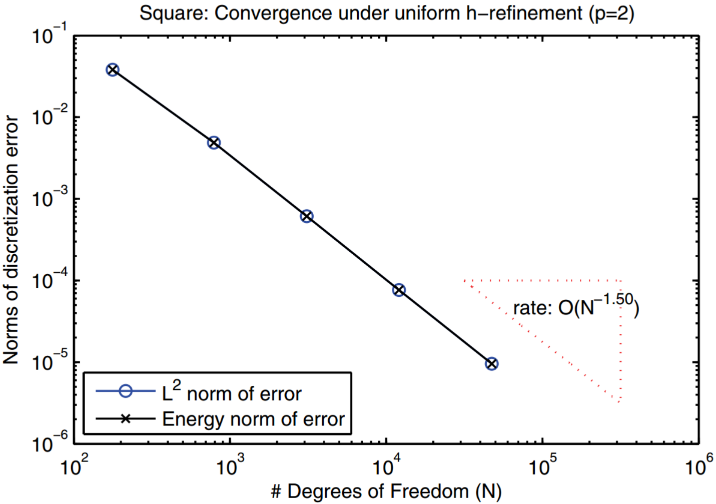


Figure: Square Case

Uniform h-refinement

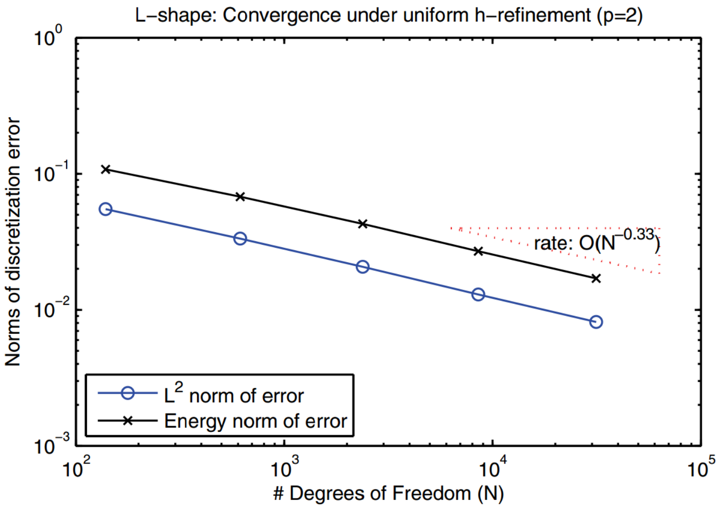
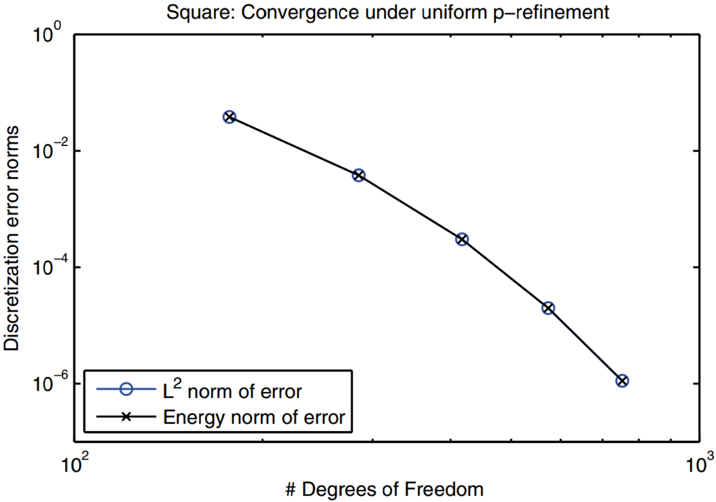
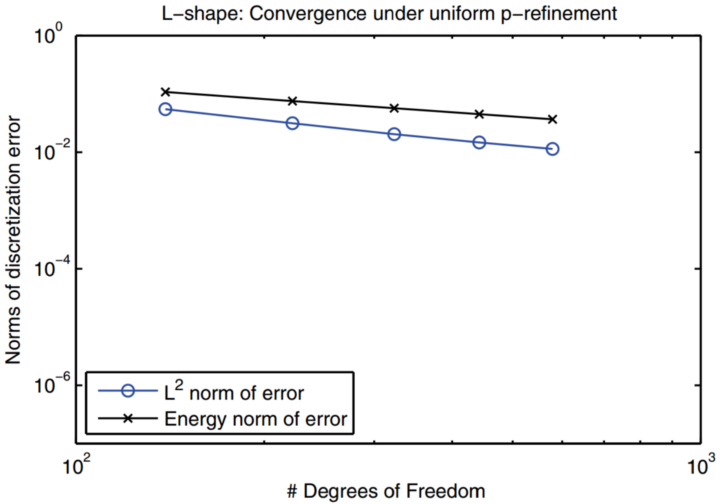


Figure: L-Shape Case

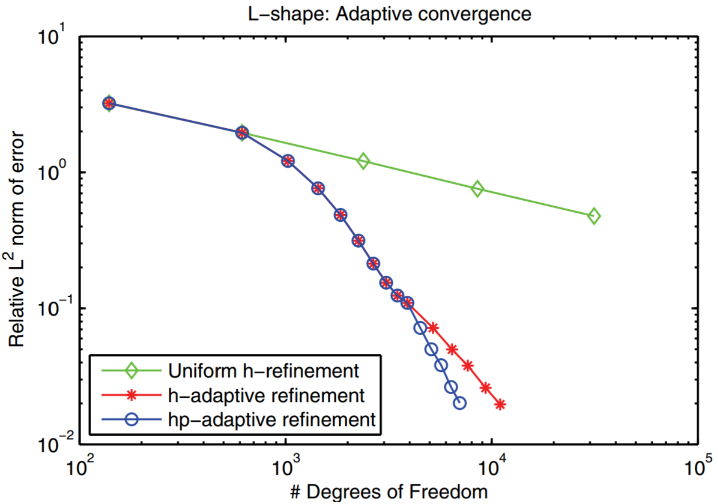
Uniform p-refinement



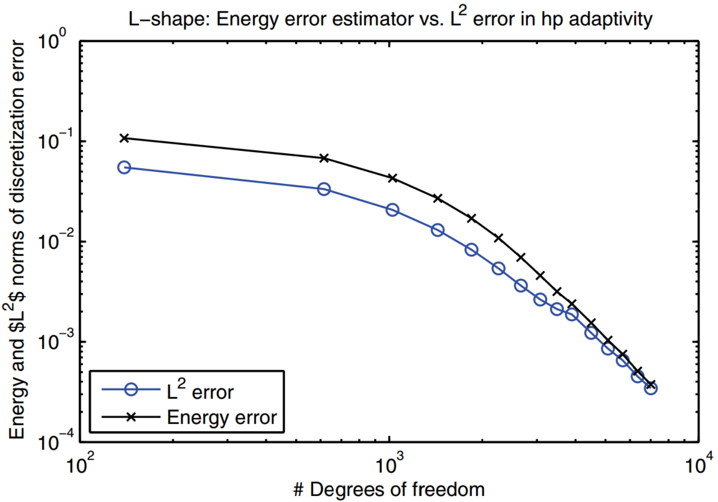
Uniform p-refinement



Adaptive refinement



Error Estimator



Effect of trace degrees

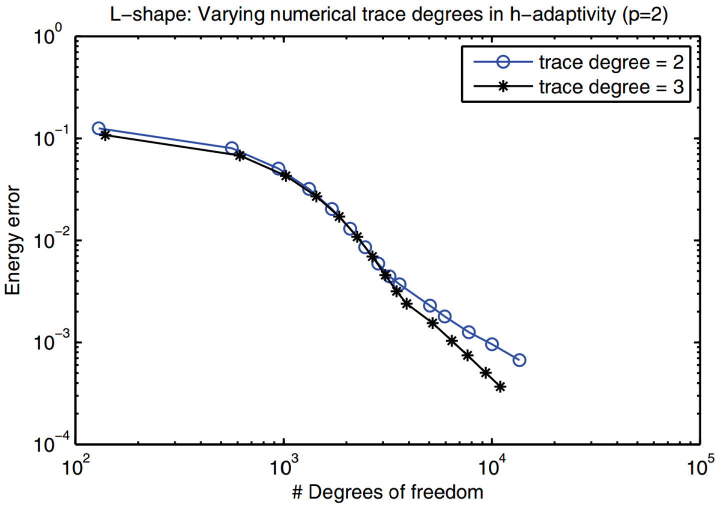


Figure: Choosing inappropriate trace degree impacts approximation

Literature

- L. Demkowicz and J. Gopalakrishnan. Analysis of the DPG method for the Poisson equation. SIAM J. Numer. Anal., 49(5):1788-1809, 2011.

Thank you for your attention!