A spacetime DPG method for acoustic waves

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Overview

1. The transient wave problem
2. The broken weak formulation
3. Verification of the density condition
4. The ideal DPG method and a priori & a posteriori error estimates
5. Implementation of practical DPG and numerical results

Outline

Problem formulation

The wave equation

Let $\Omega_0 \subset \mathbb{R}^d$ be the spatial domain with boundary $\partial\Omega_0$ and let $\Omega = \Omega_0 \times (0, T)$ be the spacetime cylinder, with $T > 0$.
Furthermore, let $f \in L^2(\Omega)$ and $g \in L^2(\Omega)^d$.
Then the considered first order system for the wave equation is given by

$$\partial_t q - grad_x \mu = g,$$
$$\partial_t \mu - div_x q = f.$$

Additionally, the wave equation is equipped with homogeneous initial and boundary conditions, i.e.,

$$\mu|_{t=0} = 0, \quad q|_{t=0} = 0, \quad \mu|_{\partial\Omega_0 \times (0,T)} = 0.$$

The formal wave operator

- Wave operator can be seen as first order distributional derivative operator

$$A : L^2(\Omega)^{d+1} \to \mathcal{D}'(\Omega)^{d+1}$$
$$Au = \begin{bmatrix} \partial_t u_q - grad_x u_\mu \\ \partial_t u_\mu - div_x u_q \end{bmatrix},$$

where $u \in L^2(\Omega)^{d+1}$ is split into

$$u = \begin{bmatrix} u_q \\ u_\mu \end{bmatrix}$$

with $u_q \in L^2(\Omega)^d$ and $u_\mu \in L^2(\Omega)$.

The formal wave operator

Theorem

Let Ω be open. The space

$$W(\Omega) := \{u \in L^2(\Omega)^{d+1} : Au \in L^2(\Omega)^{d+1}\} = W$$

endowed with the norm

$$\|u\|_W = (\|u\|^2 + \|Au\|^2)^{1/2}$$

is a Hilbert space.

Proof.

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The formal wave operator

- The formal adjoint of A is the operator $-A$ and satisfies

$$(Au, v) = -(u, Av) \qquad \text{for all } u, v \in \mathcal{D}(\Omega)^{d+1}.$$

- Furthermore, we introduce $D : W \to W'$ by

$$\langle Du, v\rangle_W = (Au, v) + (u, Av) \qquad \text{for all } u, v \in W,$$

  where W' denotes the dual space of W, and $\langle \cdot, \cdot \rangle_W$ denotes the duality pairing in W.
- Assume $u, v \in \mathcal{D}(\overline{\Omega})^{d+1}$, then

$$\langle Du, v\rangle_W = \int_{\partial\Omega} u_q \cdot (n_t v_q - n_x v_\mu) + u_\mu (n_t v_\mu - n_x \cdot v_q) ds,$$

  where $n = (n_x, n_t)$ is the outward unit normal to $\Omega \subset \mathbb{R}^{d+1}$

The unbounded wave operator

- Definition of an unbounded operator, again denoted by A
- Domain dom(A) of A takes initial and boundary conditions into account
- We partition the spacetime boundary $\partial\Omega$ into

$$\begin{aligned} \Gamma_0 &= \Omega_0 \times \{0\}, \\ \Gamma_T &= \Omega_0 \times \{T\}, \\ \Gamma_b &= \partial\Omega_0 \times [0, T], \end{aligned}$$

- Moreover, we define

$$\begin{aligned} \mathcal{V} &= \{v \in \mathcal{D}(\overline{\Omega})^{d+1} : v|_{\Gamma_0} = 0, v_\mu|_{\Gamma_b} = 0\}, \\ \mathcal{V}^* &= \{v \in \mathcal{D}(\overline{\Omega})^{d+1} : v|_{\Gamma_T} = 0, v_\mu|_{\Gamma_b} = 0\}. \end{aligned}$$

The unbounded wave operator

- Now we can define the unbounded operator

$$A : dom(A) \subset L^2(\Omega)^{d+1} \to L^2(\Omega)^{d+1},$$
$$Au = \begin{bmatrix} \partial_t u_q - grad_x u_\mu \\ \partial_t u_\mu - div_x u_q \end{bmatrix},$$

with

$$dom(A) = \{u \in W : \langle Du, v\rangle_W = 0 \text{ for all } v \in \mathcal{V}^*\}.$$

- $\mathcal{D}(\Omega)^{d+1} \subset dom(A) \subset W \implies A$ is densly defined and has an adjoint A^*
- A^* equals $-A$ when applied to

$$dom(A^*) = \{v \in L^2(\Omega)^{d+1} : \exists \ell \in L^2(\Omega)^{d+1} \text{ s. t. } (Au, v) = (u, \ell) \\ \text{for all } u \in dom(A)\}$$

The unbounded wave operator

- Throughout we denote $dom(A)$ and $dom(A^*)$ endowed with the topology of W as V and V^*, respectively
- V and V^* are closed subsets of $W \implies A, A^*$ are closed operators
- We have

$$V^* =^{\perp} D(V) := \{w \in W : \langle f, w \rangle_W = 0 \text{ for all } f \in D(V)\}$$

as well as the inclusions

$$\mathcal{V} \subset V,$$
$$\mathcal{V}^* \subset V^*.$$

Outline

The broken weak formulation

- Partition of Ω into a mesh Ω_h of open elements K with

$$\overline{\Omega} = \bigcup_{K \in \Omega_h} \overline{K}$$

- "Broken" analogue W_h of W given by

$$W_h = \{v \in L^2(\Omega)^{d+1} : A_h v \in L^2(\Omega)^{d+1}\}.$$

- Let A_h be the wave operator applied element by element, i.e.,

$$(A_h v)|_K = A(v|_K), \qquad v \in W(K), K \in \Omega_h$$

- The operator $D_h : W_h \to W_h'$ is defined by

$$\langle D_h u, v \rangle_h := \langle D_h u, v \rangle_{W_h} = (A_h u, v) + (u, A_h v)$$

for all $u, v \in W_h$ and $\langle \cdot, \cdot \rangle_h$ denotes the duality pairing in W_h

The broken weak formulation

- Let $D_{h,V}$ denote the restriction to V, i.e.,

$$D_{h,V} = D_h|_V$$

  and we define

$$Q := \mathcal{R}(D_{h,V})$$

- Q with

$$\|q\|_Q = \inf_{v \in D_{h,V}^{-1}(\{q\})} \|v\|_W$$

  is a complete space.

- The bilinear form

$$b((v,\rho),w) = -(v, A_h w) + \langle \rho, w \rangle_h$$

on $(L^2(\Omega)^{d+1} \times Q) \times W_h$ leads to

Broken weak formulation

Let $F \in W_h'$. Find: $u \in L^2(\Omega)^{d+1}$ and $\lambda \in Q$ such that

$$b((u,\lambda),w) = F(w) \text{ for all } w \in W_h. \tag{1}$$

The broken weak formulation

- Well-posedness of (1) is guaranteed, if

$$V =^{\perp} D(V^*), \tag{2}$$

$$A : V \to L^2(\Omega)^{d+1} \text{ is a bijection.} \tag{3}$$

Theorem

Suppose

$$\mathcal{V} \textit{ is dense in } V \textit{ and } \mathcal{V}^* \textit{ is dense in } V^*.$$

Then the conditions (2) and (3) are satisfied.

Proof.

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Outline

Verification of the density condition

- Application of the density result for a hyperrectangle, i.e.,

$$\Omega = \Omega_0 \times (0, T), \Omega_0 = \prod_{i=1}^{d} (0, a_i),$$

for some $a_i > 0$.

Theorem

On the previously defined Ω, $\mathcal{V}^*$ is dense in V^* and $\mathcal{V}$ is dense in V.

Verification of the density condition

Proof.

The proof is divided into three steps.

1 Extension:
Here, we extend a function in V using spatial reflections to a domain which has larger spatial extent than Ω. The operations

$$R_{i,-}x = x - 2x_i e_i, \qquad R_{i,+}x = x + 2(a_i - x_i)e_i$$

perform reflections of x about $x_i = 0$ and $x_i = a_i$ for $i = 1, \ldots, d$. The extended domains Q_i are obtained recursively by

$$\begin{aligned} Q_0 &= \overline{\Omega}, \\ Q_{i,-} = R_{i,-}^{-1} Q_{i-1}, \quad Q_{i,+} &= R_{i,+}^{-1} Q_{i-1}, \\ Q_i &= Q_{i,-} \cup Q_{i-1} \cup Q_{i,+}. \end{aligned}$$

The final extended domain is $Q = Q_d$.

Verification of the density condition

Then we need even and odd extensions of functions. Let $G_{i,e}, G_{i,o} : L^2(Q_{i-1}) \to L^2(Q_i)$ be defined by

$$G_{i,e}f(x,t) = \begin{cases} f(R_{i,-}x,t) & \text{if } (x,t) \in Q_{i,-}, \\ f(R_{i,+}x,t) & \text{if } (x,t) \in Q_{i,+}, \\ f(x,t) & \text{if } (x,t) \in Q_{i-1}. \end{cases}$$

$$G_{i,o}f(x,t) = \begin{cases} -f(R_{i,-}x,t) & \text{if } (x,t) \in Q_{i,-}, \\ -f(R_{i,+}x,t) & \text{if } (x,t) \in Q_{i,+}, \\ f(x,t) & \text{if } (x,t) \in Q_{i-1}. \end{cases}$$

For vector valued functions $v \in L^2(\Omega)^{d+1}$, we define

$$G_i v(x,t) = (G_{i,e}v_i)e_i + \sum_{j \neq i} (G_{i,o}v_j)e_j.$$

Verification of the density condition

Next, we define

$$E_k = G_k \circ G_{k-1} \circ \cdots \circ G_1,$$
$$E'_k = G'_k \circ G'_{k+1} \circ \cdots \circ G'_d,$$

where

$$G'_i w(x,t) = (G'_{i,e} w_i) e_i + \sum_{j \neq i} (G'_{i,o} w_j) e_j.$$

with

$$G'_{i,o} w(x,t) = w(x,t) - w(R_{i,-}^{-1} x) - w(R_{i,+}^{-1} x),$$
$$G'_{i,e} w(x,t) = w(x,t) + w(R_{i,-}^{-1} x) + w(R_{i,+}^{-1} x).$$

Verification of the density condition

It holds that

$$(Ev, w)_Q = (v, E'w) \qquad \text{for all } v \in L^2(\Omega)^{d+1}, w \in L^2(Q)^{d+1}.$$

It can be proven that for any $v \in V$, $AEv \in L^2(Q)^{d+1}$, AEv coincides with EAv and $Ev \in W(Q)$

Verification of the density condition

2 Translation: In this step we translate up the previously obtained extension in time coordinate. Let $v \in V$ and $\tilde{E}v$ be the extension of Ev by 0 to $\mathbb{R}^{d+1}$. For $\tau_\delta, \delta > 0$, the translation operator in time direction, i.e., $(\tau_\delta w)(x,t) = w(x, t-\delta)$ it holds that

$$\lim_{\delta \to 0} \|\tau_\delta g - g\|_{L^2(\mathbb{R}^{d+1})} = 0 \qquad \text{for all } g \in L^2(\mathbb{R}^{d+1}).$$

With the restriction H_δ from $\mathbb{R}^{d+1}$ to $Q_\delta = \prod_{i=1}^{d}(-a_i, 2a_i) \times (-\delta, T+\delta)$ it must be verified that

$$AH_\delta \tau_\delta \tilde{E}v = H_\delta \tau_\delta \tilde{E}v.$$

In particular we have $H_\delta \tau_\delta \tilde{E}v \in W(Q_\delta)$ whenever $v \in V$.

Verification of the density condition

3 Mollification: In this step we consider a $v \in V$ and mollify the time-translated extension $\tau_\delta \tilde{E} v$. The used mollifier is given by

$$\rho_\varepsilon(x,t) = \varepsilon^{-(d+1)} \rho_1(\varepsilon^{-1}x, \varepsilon^{-1}t),$$

where

$$\rho_1(x,t) = \begin{cases} k \; exp(-\frac{1}{1-|x|^2-t^2}) & \text{if } |x|^2 + t^2 < 1, \\ 0 & \text{if } |x|^2 + t^2 \geq 1 \end{cases}$$

with k such that $\int_{\mathbb{R}^{d+1}} \rho_1 = 1$. To end this proof it suffices to show that

$$v_\varepsilon = \rho_\varepsilon * \tau_\delta \tilde{E} v$$

is in $\mathcal{V}$ and $\|v - v_\varepsilon|_\Omega\|_W \xrightarrow[\varepsilon \to 0]{} 0$.

Outline

The ideal DPG method and a priori & a posteriori error estimates

- Approximation of the broken weak formulation by ideal DPG method
- Find: $u_h \in U_h \subset L^2(\Omega)^{d+1}$ and $\lambda_h \in Q_h \subset Q$ such that

$$b((u_h, \lambda_h), w_h) = F(w_h) \text{ for all } w_h \in T(U_h \times Q_h), \tag{4}$$

where $T : L^2(\Omega)^{d+1} \times Q \to W_h$ is such that

$$(T(v, \rho), w)_h = b((v, \rho), w)$$

for all $w \in W_h$ and any $(v, \rho) \in L^2(\Omega)^{d+1} \times Q$.

- The mixed formulation
  Find: $\varepsilon_h \in W_h$ and $(u_h, \lambda_h) \in (U_h \times Q_h)$ such that

$$\begin{aligned} (\varepsilon_h, w)_h + b((u_h, \lambda_h), w) &= F(w) \qquad \text{for all } w \in W_h, \\ b((v, \rho), \varepsilon_h) &= 0 \qquad \text{for all } (v, \rho) \in U_h \times Q_h \end{aligned}$$

is equivalent to formulation (4), see Seminar 08.

The ideal DPG method and a posteriori error estimates

- The expression

$$\eta = \|\varepsilon_h\|_{W_h} = \left(\sum_{K \in \Omega_h} \|\varepsilon_h\|^2_{W(K)} \right)^{1/2}$$

is an efficient and reliable a posteriori error estimator, see Seminar 08.

The ideal DPG method and a priori error estimates

- For a priori estimates we distinguish between
  Case A:: Ω_h is a geometrically conforming mesh of (d+1)-simplices

  $$V_h = \{u \in V \cap C(\overline{\Omega})^{d+1} : u|_K \in P_{p+1}(K)^{d+1} \text{ for all } K \in \Omega_h\},$$
  $$U_h = \{u \in L^2(\Omega)^{d+1} : u|_K \in P_p(K)^{d+1} \text{ for all } K \in \Omega_h\},$$

  Case B:: Ω_h is a geometrically conforming mesh of hyperrectangles

  $$V_h = \{u \in V \cap C(\overline{\Omega})^{d+1} : u|_K \in Q_{p+1}(K)^{d+1} \text{ for all } K \in \Omega_h\},$$
  $$U_h = \{u \in L^2(\Omega)^{d+1} : u|_K \in Q_p(K)^{d+1} \text{ for all } K \in \Omega_h\},$$

  where $P_p(K)$ and $Q_p(K)$ denote spaces of polynomials of total degree $\leq p$, and of degree at most p in each variable, respectively.
- $Q_h = D_h V_h$

The ideal DPG method and a priori error estimates

Theorem

Let $u \in V \cap H^{s+1}(\Omega)^{d+1}$ *and* $\lambda = D_h u$ *solve (1). Let* U_h *and* V_h *be as in one of the previously introduced cases depending on the mesh type, and* $Q_h = D_h V_h$*. Then*

$$\|u - u_h\| + \|\lambda - \lambda_h\|_Q \leq c h^s |u|_{H^{s+1}(\Omega)^{d+1}}$$

for $(d-1)/2 < s \leq p+1$*.*

Proof.

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Outline

- W_h replaced by Y_h^r
  Case A:: $Y_h^r = \{w \in W_h(\Omega) : w|_K \in P_r(K)^{d+1}\}$,
  Case B:: $Y_h^r = \{w \in W_h(\Omega) : w|_K \in Q_r(K)^{d+1}\}$.
- Considered the mixed formulation
  Find: $\varepsilon_h \in Y_h^r$ and $(u_h, \lambda_h) \in (U_h \times Q_h)$ such that

$$\begin{aligned} (\varepsilon_h, w)_h + b((u_h, \lambda_h), w) &= F(w) \qquad \text{for all } w \in Y_h^r, \\ b((v, \rho), \varepsilon_h) &= 0 \qquad \text{for all } (v, \rho) \in U_h \times Q_h \end{aligned} \tag{5}$$

- $r = p + d + 1$

Implementation of practical DPG and numerical results

- For the implementation of (5), $\lambda_h = D_h z_h$ for some $z_h \in V_h$ and

$$\begin{aligned} (\varepsilon_h, w)_h + b((u_h, D_h z_h), w) &= F(w) \qquad \text{for all } w \in Y_h^r, \\ b((v, D_h r), \varepsilon_h) &= 0 \qquad \text{for all } (v, r) \in U_h \times V_h \end{aligned} \tag{6}$$

  is considered.
- Decomposition of V_h into $V_h^0 = \{z \in V_h : z|_{\partial K} = 0 \text{ for all } K \in \Omega_h\}$ and remainder $V_h^1 = V_h \backslash V_h^0$.
- $b((v, D_h V_h^0), w) = 0$, thus replace V_h by V_h^1 in (6).

- Yields the matrix equation

$$\begin{bmatrix} A & B \\ B^\top & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{7}$$

  where e and x are the vectors of coefficients in the basis expansion of $\varepsilon_h \in Y_h^r$ and $(u_h, z_h) \in U_h \times V_h$, respectively, $[A]_{kl} = (y_l, y_k)_h$, $[B_0]_{ki} = b((u_i, 0), y_k)$, $[B_1] = b((0, D_h z_j), y_k)$ and $B = [B_0, B_1]$.

- $r = p + d + 1$
- $\mathcal{N}(A) = \mathcal{N}(B_0) = \{0\}$
- B_1 may have a nontrivial kernel

Techniques to solve despite the null space

- **Technique 1**: Remaining orthogonal to null space in conjugate gradients
  - Instead of (7) solve Schur complement system

    $$\underbrace{B^\top A^{-1} B}_{=:C} x = B^\top A^{-1} f$$

    by means of CG
  - $\mathit{ker}\ C = \mathit{ker}\ B = \mathit{ker}\ B_1$
  - Convergence if $K_n(C, r_0)$ remains ℓ^2 orthogonal to $\mathit{ker}\ C$ for all n
  - $x_0 = 0 \Longrightarrow r_0 = B^\top A^{-1} f \in \mathcal{R}(B^\top) = (\mathit{ker}\ C)^\perp$
  - $C^n r_0$ is orthogonal to $\mathit{ker}\ C$ for all $n \geq 1$

Techniques to solve despite the null space

- **Technique 2**: Regularization of the linear system
  - Rewriting $B^\top A^{-1} B x = B^\top A^{-1} f$ in block form yields

$$\begin{bmatrix} B_0^\top A^{-1} B_0 & B_0^\top A^{-1} B_1 \\ B_1^\top A^{-1} B_0 & B_1^\top A^{-1} B_1 \end{bmatrix} x = B^\top A^{-1} f$$

  - Solving the invertible system

$$\begin{bmatrix} B_0^\top A^{-1} B_0 & B_0^\top A^{-1} B_1 \\ B_1^\top A^{-1} B_0 & B_1^\top A^{-1} B_1 + \alpha M \end{bmatrix} x = B^\top A^{-1} f$$

  with the mass matrix $M_{jl} = (z_l, z_j)$ and $\alpha > 0$, e.g., $\alpha = 10^{-9}$.

Convergence rates in two-dimensional spacetime

- $\Omega = (0,1)^2$, homogeneous boundary and initial conditions, the exact solution of the second order wave equation is given by

$$\phi(x,t) = sin(\pi x) sin^2(\pi t)$$

which results in a solution

$$u = \begin{bmatrix} \pi cos(\pi x) sin^2(\pi t) \\ \pi sin(\pi x) sin(2\pi t) \end{bmatrix}$$

of the first order system

- $g = 0, f = \pi^2 sin(\pi x)(2cos(2\pi t) + sin^2(\pi t))$

Convergence rates in two-dimensional spacetime

h	$p=0$	Order	$p=1$	Order	$p=2$	Order	$p=3$	Order
1/4	1.2849e+00	–	1.5371e-01	–	2.0385e-02	–	1.2619e-03	–
1/8	5.6379e-01	1.19	5.6127e-02	1.45	4.7540e-03	2.10	1.5370e-04	+3.04
1/16	2.2067e-01	1.35	1.2472e-02	2.17	5.4897e-04	3.11	7.8519e-06	+4.29
1/32	1.0214e-01	1.11	3.0308e-03	2.04	6.6955e-05	3.00	4.7863e-07	+4.04

Table 1. Convergence rates for $\|u - u_h\|$ on triangular meshes using Technique 1.

h	$p=0$	Order	$p=1$	Order	$p=2$	Order	$p=3$	Order
1/4	9.7226e-01	–	1.6834e-01	–	6.6722e-03	–	2.0910e-03	–
1/8	4.7357e-01	1.04	4.2869e-02	1.97	8.5059e-04	2.97	1.3308e-04	3.97
1/16	2.3291e-01	1.35	1.0763e-02	1.99	1.0707e-04	2.99	8.3773e-06	3.99
1/32	1.1587e-01	1.11	2.6935e-03	2.00	1.3409e-05	3.00	5.2613e-07	3.99

Table 2. Convergence rates for $\|u - u_h\|$ on rectangular meshes using Technique 1.

Figure: L^2 convergence rates for u_h, taken from [J. Gopalakrishnan and P. Sepúlveda, (2017)]

Adaptivity

- $\Omega = (0,1)^2$, $f = g = 0$, homogeneous boundary conditions, i.e., $\mu = 0$ and non-zero initial conditions

$$\mu|_{t=0} = -\phi_0, q|_{t=0} = \phi_0$$

with

$$\phi_0(x) = exp(-1000((x - 0.5)^2))$$

Adaptivity

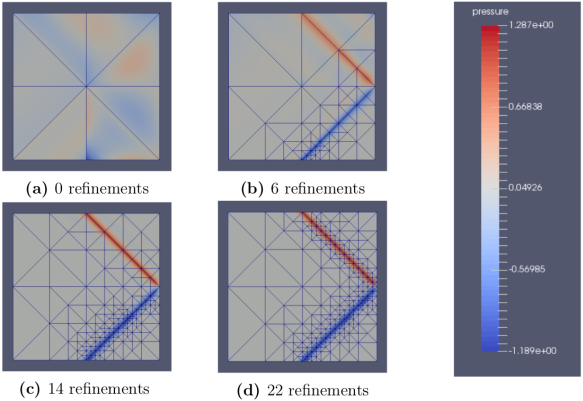

Figure: Numerical pressure μ, adaptive refinement, setting p = 3, taken from [J. Gopalakrishnan and P. Sepúlveda, (2017)]

Convergence rates in three-dimensional spacetime

- $\Omega = (0,1)^3$, homogeneous boundary and initial conditions, the exact solution of the second order wave equation is given by

  $$\phi(x,t) = sin(\pi x) sin(\pi y) t^2$$

  which results in a solution

  $$u = \begin{bmatrix} \pi cos(\pi x) sin(\pi y) t^2 \\ \pi cos(\pi y) sin(2\pi x) t^2 \\ 2 sin(\pi x) sin(\pi y) t \end{bmatrix}$$

  of the first order system
- $g = 0, f = sin(\pi x) sin(\pi y)(2 + 2\pi^2 t^2)$

Convergence rates in three-dimensional spacetime

h	$p=0$	Order	$p=1$	Order	$p=2$	Order	$p=3$	Order
1	9.0604e-01	–	4.7829e-01	–	1.4146e-01	–	4.3952e-02	–
1/2	6.0557e-01	0.58	1.3924e-01	1.78	1.3912e-02	3.35	3.2845e-03	3.74
1/4	3.3896e-01	0.84	3.3508e-02	2.05	1.4769e-03	3.24	1.6490e-04	4.32
1/8	1.5469e-01	1.13	8.9554e-03	1.90	1.7210e-04	3.10	9.9691e-06	4.05

Table 3. Convergence rates for $\|u - u_h\|$ on tetrahedral meshes obtained using Technique 2.

h	$p=0$	Order	$p=1$	Order	$p=2$	Order	$p=3$	Order
1	1.1149e+00	–	6.0068e-01	–	2.8828e-02	–	3.3262e-02	-
1/2	7.5769e-01	0.56	1.5124e-01	1.99	2.8264e-03	3.35	2.0540e-03	4.02
1/4	4.2035e-01	0.85	3.8592e-02	1.97	3.5256e-04	3.00	1.3234e-04	3.96
1/8	2.1338e-01	0.98	9.6918e-03	1.99	3.8023e-05	3.21	9.3766e-06	3.82

Table 4. Convergence rates for $\|u - u_h\|$ on hexahedral meshes using Technique 1.

Figure: L^2 convergence rates for u_h, taken from [J. Gopalakrishnan and P. Sepúlveda, (2017)]

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