Robust DPG method for convection-dominated diffusion problem

Seminar on Numerical Analysis
Discontinuous Petrov-Galerkin (DPG) Methods

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Outline

References

[DG11] L. Demkowicz and J. Gopalakrishnan, *Analysis of the DPG method for the Poisson equation*, SIAM J. Numer. Anal. **49** (2011), no. 5, 1788–1809.

[DH13] L. Demkowicz and N. Heuer, *Robust DPG method for convection-dominated diffusion problems*, SIAM J. Numer. Anal. **51** (2013), no. 5, 2514–2537.

- Introduction
- An extension to the convection diffusion equation
- Robust DPG Method
- Numerical examples
- Conclusion

A model problem

We consider the convection-diffusion ("confusion") problem

$$\begin{aligned} -\alpha\Delta u + \nabla\cdot(\boldsymbol{\beta} u) &= f && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

with its first order reformulation

$$\begin{aligned} \alpha^{-1}\boldsymbol{\sigma} - \nabla u &= 0 && \text{in } \Omega,\\ -\nabla\cdot\boldsymbol{\sigma} + \nabla\cdot(\boldsymbol{\beta} u) &= f && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

Special case $\nabla \cdot \boldsymbol{\beta} = 0$

Strong form

Note that $\nabla \cdot (\boldsymbol{\beta} u) = \boldsymbol{\beta} \cdot \nabla u + (\nabla \cdot \boldsymbol{\beta})u$, hence, for $\nabla \cdot \boldsymbol{\beta} = 0$, we obtain

$$
\begin{aligned}
\alpha^{-1}\boldsymbol{\sigma} - \nabla u &= 0 && \text{in } \Omega, \\
-\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{\beta} \cdot \nabla u &= f && \text{in } \Omega, \\
u &= 0 && \text{on } \Gamma.
\end{aligned}
$$

Special case $\nabla \cdot \boldsymbol{\beta} = 0$

Ultra-weak form

Find $(u, \boldsymbol{\sigma}, \hat{u}, \hat{\sigma}_n) \in U$, s.t.

$$b((u, \boldsymbol{\sigma}, \hat{u}, \hat{\sigma}_n), (v, \boldsymbol{\tau})) = l(v, \boldsymbol{\tau}),$$

with

$$\begin{aligned} b((u, \boldsymbol{\sigma}, \hat{u}, \hat{\sigma}_n), (\boldsymbol{\tau}, v)) =& (\alpha^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau})_{\Omega_h} + (u, \nabla \cdot \boldsymbol{\tau})_{\Omega_h} - \langle [\boldsymbol{\tau} \cdot \boldsymbol{n}], \hat{u} \rangle \\ &+ (\boldsymbol{\sigma} - \boldsymbol{\beta} u, \nabla v)_{\Omega_h} - \langle \hat{\sigma}_n, [v] \rangle \end{aligned}$$

where

$$\hat{\sigma}_n = ((\boldsymbol{\sigma} - \boldsymbol{\beta} u) \cdot \boldsymbol{n})|_{\Gamma_h}$$

Some spaces & norms I

In the following, we need the spaces

$$U := L_2(\Omega) \times \boldsymbol{L}_2(\Omega) \times H_0^{1/2}(\Gamma_h) \times H^{-1/2}(\Gamma_h),$$
$$V := H^1(\Omega_h) \times \boldsymbol{H}(\mathrm{div}, \Omega_h),$$

where

$$\begin{aligned} H^1(\Omega_h) &:= \{v \in L_2(\Omega) : v|_K \in H^1(K) \quad \forall K \in \Omega_h\}, \\ H_0^1(\Omega_h) &:= \{v \in H^1(\Omega_h) : v|_\Gamma = 0\}, \\ \boldsymbol{H}(\mathrm{div}, \Omega_h) &:= \{\boldsymbol{\tau} \in \boldsymbol{L}_2(\Omega) : \boldsymbol{\tau}|_K \in \boldsymbol{H}(\mathrm{div}, K) \quad \forall K \in \Omega_h\}, \\ H_0^{1/2}(\Gamma_h) &:= H_0^1(\Omega)|_{\Gamma_h}, \\ H^{-1/2}(\Gamma_h) &:= \{\eta : \exists \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, \Omega) : (\boldsymbol{\tau} \cdot \boldsymbol{n})|_{\Gamma_h} = \eta\}. \end{aligned}$$

Some spaces & norms II

Moreover, we need

$$\|v\|_{1/2,\Gamma_h} := \inf_{w \in H_0^1(\Omega), w|_{\Gamma_h} = v} \|w\|_{1,\Omega},$$

$$\|\eta\|_{-1/2,\Gamma_h} := \inf_{\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div},\Omega), (\boldsymbol{\tau}\cdot\boldsymbol{n})|_{\Gamma_h}} \|\boldsymbol{\tau}\|_{\boldsymbol{H}(\mathrm{div},\Omega)},$$

and

$$\|[\boldsymbol{\tau}\cdot\boldsymbol{n}]\|_{\Gamma_h^0} := \sup_{w \in H_0^{1/2}(\Gamma_h)} \frac{\langle [\boldsymbol{\tau}\cdot\boldsymbol{n}], w\rangle}{\|w\|_{1/2,\Gamma_h}} = \sup_{w \in H_0^1(\Omega)} \frac{\langle [\boldsymbol{\tau}\cdot\boldsymbol{n}], w\rangle}{\|w\|_{1,\Omega}}$$

$$\|[v]\|_{\Gamma_h} := \sup_{\eta \in H^{-1/2}(\Gamma_h)} \frac{\langle \eta, [v]\rangle}{\|\eta\|_{-1/2,\Gamma_h}} = \sup_{\boldsymbol{\eta} \in \boldsymbol{H}(\mathrm{div},\Omega)} \frac{\langle \boldsymbol{\eta}\cdot\boldsymbol{n}, [v]\rangle}{\|\boldsymbol{\eta}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}}$$

The optimal test norm

Let U be the trialspace, equipped with the norm $\|\cdot\|_U$. Then the optimal test norm in the DPG frame work is defined as

$$\begin{aligned}\|(v,\boldsymbol{\tau})\|_{V,\mathrm{opt}}^2 := \sup_{(u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n)\in U} \frac{b((u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n),(v,\boldsymbol{\tau}))}{\|(u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n)\|_U} \\ =\|\alpha^{-1}\boldsymbol{\tau}+\nabla v\|_{\Omega_h}^2 + \|\nabla\cdot(\boldsymbol{\tau}-\boldsymbol{\beta} v)\|_{\Omega_h}^2 \\ + \|[v]\|_{\Gamma_h}^2 + \|[\boldsymbol{\tau}\cdot\boldsymbol{n}]\|_{\Gamma_h^0}^2 .\end{aligned}$$

An abstract result

Theorem (Abstract theorem, [DG11])

Let u and u_h be the exact and approximate solution, respectively. Assume that $b(\cdot,\cdot)$ is injective and that there are positive constants C_1, C_2 such that

$$C_1\|(v,\boldsymbol{\tau})\|_V \leq \|(v,\boldsymbol{\tau})\|_{V,\mathrm{opt}} \leq C_2\|(v,\boldsymbol{\tau})\|_V \qquad \forall v \in V.$$

Then

$$\|u-u_h\|_U \leq \frac{C_2}{C_1} \inf_{w_h\in U_h} \|u-w_h\|_U.$$

For the lower bound, we need two Lemmata.

Lemma ([DG11])

Let $\boldsymbol{\tau}_0 \in \boldsymbol{H}(\mathrm{div}, \Omega_h)$ *and* $v_0 \in H^1(\Omega_h)$ *satisfy*

$$\begin{aligned} \alpha^{-1}\boldsymbol{\tau}_0 - \nabla v_0 &= 0 && \text{on } K \\ \nabla \cdot (\boldsymbol{\tau}_0 - \boldsymbol{\beta} v_0) &= 0 && \text{on } K \end{aligned}$$

for every element K *in* Ω_h*. Then*

$$\|\boldsymbol{\tau}_0\| + \|\nabla v_0\|_{\Omega_h} \leq C(\alpha, \boldsymbol{\beta})\big(\|[\boldsymbol{\tau}_0 \cdot \boldsymbol{n}]\|_{\Gamma_h^0} + \|[v_0]\|_{\Gamma_h}\big)$$

Proof.

On the blackboard. □

Lemma ([DG11])

Let $\boldsymbol{G} \in \boldsymbol{L}_2(\Omega =$ *and* $F \in L_2(\Omega)$. *There is a* $\boldsymbol{\tau}_1$ *in* $\boldsymbol{H}(\mathrm{div}, \Omega)$ *and* v_1 *in* $H_0^1(\Omega)$ *satisfying the adjoint problem*

$$\begin{aligned} \alpha^{-1}\boldsymbol{\tau}_1 - \nabla v_1 &= \boldsymbol{G} && \text{in } \Omega, \\ \nabla \cdot (\boldsymbol{\tau}_1 - \boldsymbol{\beta} v_1) &= F && \text{in } \Omega, \end{aligned}$$

and

$$\|\boldsymbol{\tau}_1\| + \|\nabla v_1\| \leq C(\alpha, \boldsymbol{\beta})\big(\|\boldsymbol{G}\| + \|F\|\big).$$

Proof.

See [DG11]. □

The final result

Theorem ([DG11])

Suppose u *and* u_h *be the exact and approximate solutions in* U *and* U_h*, respectively. Then there is a* $C(\alpha) > 0$ *independent of* U_h *and the partition* Ω_h*, s.t.*

$$\mathscr{D} \leq C(\alpha)\mathscr{A},$$

where

$$\mathscr{D} := \|u - u_h\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\hat{u} - \hat{u}_h\|_{1/2,\Gamma_h} + \|\hat{\sigma}_n - \hat{\sigma}_{n,h}\|_{-1/2,\Gamma_h},$$
$$\mathscr{A} := \inf_{(w_h,\rho_h,\hat{z}_h,\hat{\eta}_{m,h}) \in U_h} \|u - w_h\| + \dots,$$

are the discretization and best approximation error, respectively.

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are the discretization and best approximation error, respectively.

But what happens if $\alpha = \epsilon \ll 1$ or even $\alpha = \epsilon \to 0$?

The final result

Theorem ([DG11])

Suppose u and u_h be the exact and approximate solutions in U and U_h, respectively. Then there is a $C(\alpha) > 0$ independent of U_h and the partition Ω_h, s.t.

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are the discretization and best approximation error, respectively.

But what happens if $\alpha = \epsilon \ll 1$ or even $\alpha = \epsilon \to 0$?

We need *robust* (in $\alpha = \epsilon$) estimates!

A model problem

Recall

$$
\begin{aligned}
\epsilon^{-1}\boldsymbol{\sigma} - \nabla u &= 0 && \text{in } \Omega,\\
-\nabla \cdot \boldsymbol{\sigma} + \nabla \cdot (\boldsymbol{\beta} u) &= f && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma,
\end{aligned}
$$

and its ultra-weak formulation with the bilinear form

$$
\begin{aligned}
b((u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n),(v,\boldsymbol{\tau})) :=& (\boldsymbol{\sigma}, \epsilon^{-1}\boldsymbol{\tau} + \nabla v)_{\Omega_h} + (u, \nabla \cdot \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \nabla v)_{\Omega_h}\\
& - \langle [\boldsymbol{\tau} \cdot \boldsymbol{n}], \hat{u} \rangle - \langle \hat{\sigma}_n, [v] \rangle,
\end{aligned}
$$

with

$$
\hat{\sigma}_n = ((\boldsymbol{\sigma} - \boldsymbol{\beta} u) \cdot \boldsymbol{n})|_{\Gamma_h}.
$$

The main idea

To obtain robust (in ϵ) Cea-like estimates, we will make use of the following fact:
Robustness in the norm relation

$$\|\cdot\|_{U_1} \lesssim \|\cdot\|_E \lesssim \|\cdot\|_{U_2}$$

imply robustness in the estimates

$$\begin{aligned} \|u - u_h\|_{U_1} &\lesssim \inf_{w_h \in U_h} \|u - w_h\|_E, && (1) \\ &\lesssim \inf_{w_h \in U_h} \|u - w_h\|_{U_2}. && (2) \end{aligned}$$

Preliminaries I

A weight function

Let $\phi \in C^2(\bar{\Omega})$ be a fixed weight function, satisfying

$$0 \leq \phi \leq 1 \text{ in } \Omega, \qquad \phi = 0 \text{ on } \Gamma_-,$$

where $\Gamma_- := \{\boldsymbol{x} \in \Gamma : \boldsymbol{\beta}(\boldsymbol{x}) \cdot \boldsymbol{n} < 0\}$ is the inflow boundary.

A formal extension of $\boldsymbol{\beta}$

In any point of Ω, we extend $\boldsymbol{\beta}$ to an orthogonal basis $(\boldsymbol{\beta}, \boldsymbol{\beta}_1^\perp, \boldsymbol{\beta}_2^\perp)$ of $\mathbb{R}^3$, with

$$\|\boldsymbol{\beta}^\perp \cdot \boldsymbol{\eta}\|^2 := \|\boldsymbol{\beta}_1^\perp \cdot \boldsymbol{\eta}\|^2 + \|\boldsymbol{\beta}_2^\perp \cdot \boldsymbol{\eta}\|^2, \quad \boldsymbol{\eta} \in \boldsymbol{L}_2(\Omega).$$

Preliminaries II

Assumptions on β

We require $\boldsymbol{\beta} \in \boldsymbol{C}^2(\bar{\Omega})$ and $\boldsymbol{\beta} = O(1)$, $\nabla \cdot \boldsymbol{\beta} = O(1)$. Moreover, we might require

(A1) $\nabla \times \boldsymbol{\beta} = 0, \qquad 0 < C \leq |\boldsymbol{\beta}|^2 + \frac{1}{2}\nabla \cdot \boldsymbol{\beta}, \qquad C = O(1)$,

(A2) $\nabla(\phi\boldsymbol{\beta}) + \nabla(\phi\boldsymbol{\beta})^\top - \nabla \cdot (\phi\boldsymbol{\beta})\boldsymbol{I} = O(1)$,

(A3) $\nabla \cdot \boldsymbol{\beta} = 0$.

Preliminaries III

A dual norm

$$\|(w,\eta)\|_{\Gamma_h'} := \sup_{(v,\boldsymbol{\tau})\in V\setminus\{0\}} \frac{\langle[\boldsymbol{\tau}\cdot\boldsymbol{n}],w\rangle + \langle\eta,[v]\rangle}{\|(v,\boldsymbol{\tau})\|_{\Gamma_h}}$$

for all $(w,\eta)\in H_0^{1/2}(\Gamma_h)\times H^{-1/2}(\Gamma_h)$, with $\|(v,\boldsymbol{\tau})\|_{\Gamma_h} := \|\tilde{v}\|$. Here $(v,\boldsymbol{\tau}) = (v_0,\boldsymbol{\tau}_0) + (\tilde{v},\tilde{\boldsymbol{\tau}})$ with $(v_0,\boldsymbol{\tau}_0)\in H_0^1(\Omega)\times \boldsymbol{H}(\mathrm{div},\Omega)$ solving

$$\begin{aligned} \epsilon^{-1}\boldsymbol{\tau}_0 + \nabla v_0 &= \epsilon^{-1}\boldsymbol{\tau} + \nabla v, \\ \nabla\cdot\boldsymbol{\tau}_0 - \boldsymbol{\beta}\cdot\nabla v_0 &= \nabla\cdot\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\nabla v. \end{aligned}$$

Preliminaries IV

Some test norms

$$
\begin{aligned}
\|(v,\boldsymbol{\tau})\|_{V,\text{qopt}}^2 &:= \|v\|^2 + \|\epsilon^{-1}\boldsymbol{\tau} + \nabla v\|^2 + \|\nabla\cdot\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\nabla v\|^2,\\
\|(v,\boldsymbol{\tau})\|_{V,0} &:= \epsilon\|v\|^2 + \epsilon\|\nabla v\|^2 + \|\boldsymbol{\beta}\cdot\nabla v\|_{\phi+\epsilon}^2 + \frac{1}{\epsilon^2}\|\boldsymbol{\beta}\cdot\boldsymbol{\tau}\|_{\phi+\epsilon}^2\\
&\quad + \frac{1}{\epsilon}\|\boldsymbol{\tau}\|^2 + \|\nabla\cdot\boldsymbol{\tau}\|_{\phi+\epsilon}^2,\\
\|(v,\boldsymbol{\tau})\|_{V,1} &:= \epsilon\|v\|^2 + \epsilon\|\nabla v\|^2 + \|\boldsymbol{\beta}\cdot\nabla v\|_{\phi+\epsilon}^2 + \|\boldsymbol{\tau}\|_{\phi+\epsilon}^2\\
&\quad + \|\nabla\cdot\boldsymbol{\tau}\|_{\phi+\epsilon}^2.
\end{aligned}
$$

The trial norms

$$
\|(u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n)\|_{E,*} := \sup_{(v,\boldsymbol{\tau})\in V} \frac{b((u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n),(v,\boldsymbol{\tau}))}{\|(v,\boldsymbol{\tau})\|_{V,*}}
$$

The main result I

Theorem (DPG method with quasi-optimal test norm $\|\cdot\|_{V,\mathrm{qopt}}$, $\|\cdot\|_E = \|\cdot\|_{E,\mathrm{qopt}}$)

If β satisfies (A1), then the norm relations (1) *and* (2) *hold with*

$$\|(u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n)\|_{U_1} = \|(u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n)\|_{U_2} := \left(\|u\|^2 + \|\boldsymbol{\sigma}\|^2 + \|(\hat{u},\hat{\sigma}_n)\|^2_{\Gamma'_h}\right)^{1/2}.$$

If β satisfies (A1) and (A3), then the first norm relation (1) *holds with*

$$\|(u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n)\|_{U_1} := \left(\|u\|^2 + \|\boldsymbol{\sigma}\|^2 + \epsilon^2\|\hat{u}\|^2_{1/2,\Gamma_h} + \epsilon\|\hat{\sigma}_n\|_{-1/2,\Gamma_h}\right)^{1/2}.$$

The main result II

Theorem (DPG method with weighted test norm $\|\cdot\|_{V,0}, \|\cdot\|_E = \|\cdot\|_{E,0}$)

The second norm relation (2) *holds with*

$$\|(u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n)\|_{U_2} := \Big(\|u\|^2_{1/(\phi+\epsilon)} + \|\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\|^2_{1/(\phi+\epsilon)} + \frac{1}{\epsilon}\|\boldsymbol{\beta}^\perp\cdot\boldsymbol{\sigma}\|^2 + \frac{1}{\epsilon}\|\hat{u}\|_{1/2,\Gamma_h} + \frac{1}{\epsilon}\|\hat{\sigma}_n\|_{-1/2,\Gamma_h}\Big)^{1/2}.$$

If $\boldsymbol{\beta}$ satisfies (A1), (A2) and (A3), then the norm relation (1) *holds with*

$$\|(u,\boldsymbol{\sigma},\hat{u},\hat{\sigma}_n)\|_{U_1} := \Big(\|u\|^2 + \|\boldsymbol{\sigma}\|^2 + \epsilon^2\|\hat{u}\|^2_{1/2,\Gamma_h} + \epsilon\|\hat{\sigma}_n\|_{-1/2,\Gamma_h}\Big)^{1/2}.$$

The main result III

Theorem (DPG method with weighted test norm $\|\cdot\|_{V,1}, \|\cdot\|_E = \|\cdot\|_{E,1}$)

The second norm relation (2) *holds with*

$$\|(u, \boldsymbol{\sigma}, \hat{u}, \hat{\sigma}_n)\|_{U_2} := \Big(\|u\|^2_{1/(\phi+\epsilon)} + \frac{1}{\epsilon^2}\|\boldsymbol{\sigma}\|^2_{1/(\phi+\epsilon)} + \frac{1}{\epsilon}\|\hat{u}\|_{1/2,\Gamma_h} + \frac{1}{\epsilon}\|\hat{\sigma}_n\|_{-1/2,\Gamma_h}\Big)^{1/2}.$$

If β satisfies (A1), (A2) and (A3), then the norm relation (1) *holds with*

$$\|(u, \boldsymbol{\sigma}, \hat{u}, \hat{\sigma}_n)\|_{U_1} := \left(\|u\|^2 + \|\boldsymbol{\sigma}\|^2 + \epsilon^2\|\hat{u}\|^2_{1/2,\Gamma_h} + \epsilon\|\hat{\sigma}_n\|_{-1/2,\Gamma_h}\right)^{1/2}.$$

Proof.

Very technical, depends on the stability of the adjoint problem, see [DH13]. □

Numerical examples I

Consider the model problem

$$-\epsilon\Delta u + \boldsymbol{\beta}\cdot\nabla u = f \quad \text{in } \Omega = (0,1)^2, \qquad u = u_0 \quad \text{on } \Gamma$$

with $\boldsymbol{\beta} = \nabla(e^x \sin y)$ and f, u_0 such that

$$u(x,y) = \arctan\left(\frac{1 - |(x,y)|}{\epsilon}\right).$$

Here, (A1) and (A3) are satisfied, but not (A2). Moreover, the inflow boundary is at $x = 0$ and $y = 0$, and the weight function is $\phi(x,y) = xy/(x+y)$.

Numerical examples II

The test norm

$$\|(v,\boldsymbol{\tau})\|_{V,2}^2 := \left\|\min\left\{\frac{\epsilon}{h_1 h_2}, 1\right\} v\right\|^2 + \|\underline{\boldsymbol{\beta}} \cdot \nabla v\|_{\phi+\epsilon}^2 + \epsilon\|\underline{\boldsymbol{\beta}}^\perp \cdot \nabla v\|^2$$
$$+ \min\left\{\frac{1}{\epsilon}, \frac{1}{h_1 h_2}\right\} \|\boldsymbol{\tau}\|_{\phi+\epsilon}^2 + \|\nabla \cdot \boldsymbol{\tau}\|_{\phi+\epsilon}^2,$$

with $\underline{\boldsymbol{\beta}} := \boldsymbol{\beta}/|\boldsymbol{\beta}|$.

Numerical examples III

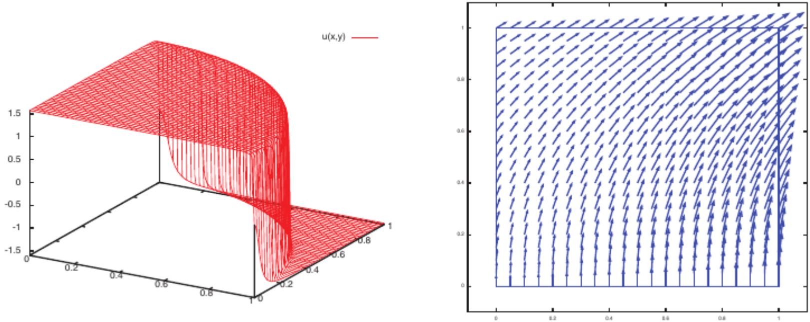

Figure: Solution u and $\boldsymbol{\beta}$, for $\epsilon = 10^{-3}$.

Numerical examples IV

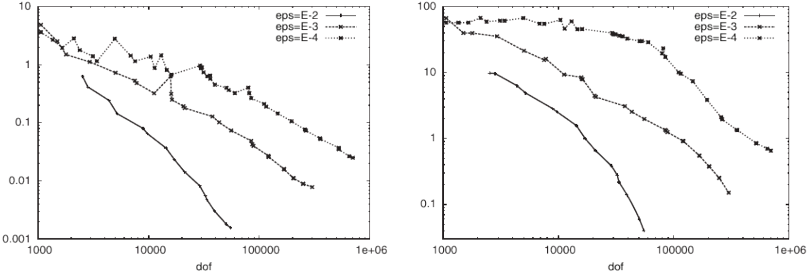

Figure: Left: convergence of the energy norm error. Right: convergence in relative L_2-norm for the fields variables.

Numerical examples V

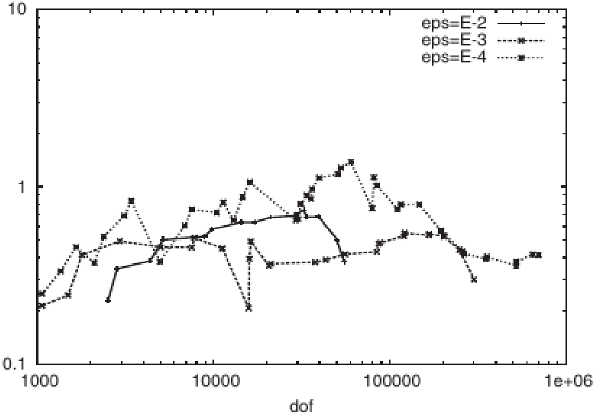

Figure: Norm ratio between L_2- and energy norm.

Numerical examples VI

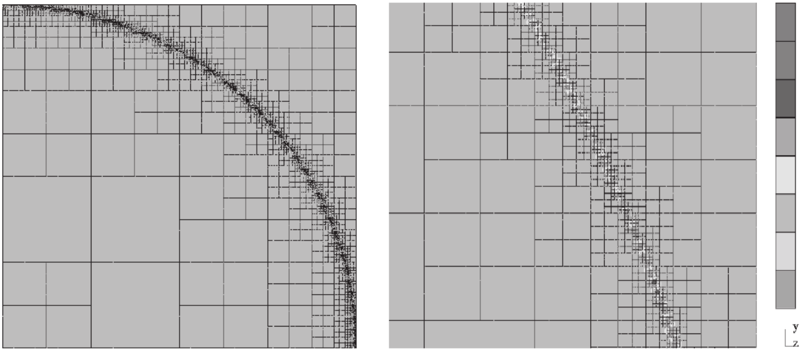

Figure: FE mesh and a zoom to the boundary layer after 35 refinement cycles.

Conclusions

- analysis of the *"confusion"* problem is a simple extension of the results for the Poisson equation,
- however, for *robustness* wrt the diffusion ϵ, a more involved analysis is required,
- with this frame work, we have robustness for three test norms,
- the quasi-optimal test norm is not practical,
- the rescaled test norm $\|\cdot\|_{V,1}$ is a *optimal* compromise between robustness and practicality.

Thank you!