DPG methods for convection dominated diffusion in 1D and 2D

Seminar on Numerical Analysis
Discontinuous Petrov-Galerkin (DPG) Methods

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Outline

1. A Model Problem
2. 1D - One Element
3. 1D - Multielement
4. Convection-Dominated Diffusion in 2D
5. Conclusions

The four basic steps

S1. Develop mesh-dependent variational formulations with an underlying space V, allowing interelement discontinuities.

S2. Choose a trial subspace U_n with good approximation properties.

S3. Approximately compute optimal test functions.

S4. Solve a symmetric positive definite (SPD) matrix system.

A convection-diffusion model problem

Classical Form

Find u s.t.,

$$\begin{aligned} -\operatorname{div}(\epsilon\,\nabla u) + \boldsymbol{\beta}\cdot\nabla u &= f && \text{in } \Omega,\\ u &= u_0 && \text{on } \partial\Omega. \end{aligned}$$

Mixed form

Find u, σ, s.t.,

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 && \text{in } \Omega,\\ -\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{\beta}\cdot\nabla u &= f && \text{in } \Omega,\\ u &= u_0 && \text{on } \partial\Omega. \end{aligned}$$

1D - One Element (Spectral Analysis)

A 1D model problem

Consider the following problem: find u, σ, such that (s.t.)

$$\begin{aligned} \frac{1}{\epsilon}\sigma - u' &= 0 \\ -\sigma' + u' &= f \\ u(0) = u_0, u(1) &= 0 \end{aligned}$$

1D - One Element (Spectral Analysis)

The variational formulation

Find $\sigma, u \in L_2(0,1)$, $\hat{\sigma}(0) \in \mathbb{R}$, $\hat{\sigma}(1)$, s.t.

$$\frac{1}{\epsilon}\int_0^1 \sigma\,\tau\;\mathrm{d}x + \int_0^1 u\,\tau'\;\mathrm{d}x = -u_0\,\tau(0)$$

$$\int_0^1 \sigma\,v'\;\mathrm{d}x - \int_0^1 u\,v'\;\mathrm{d}x + \hat{\sigma}(0)v(0) - \hat{\sigma}(1)v(1) = \int_0^1 f\,v\;\mathrm{d}x - u_0\,v(0)$$

for all $\tau, v \in H^1(0,1)$.

The choice of $(\,.\,,\,.\,)_V$

We choose $(\,.\,,\,.\,)_V$ associated to the norm

$$\|(\tau, v)\|^2 := (1-\alpha)\|\tau\|^2 + \alpha\|v\|^2,$$

with $\alpha \in (0,1)$ a scaling parameter and

$$\|\tau\|^2 = \int_0^1 |\tau'|^2 \,\mathrm{d}x + |\tau(0)|^2,$$
$$\|v\|^2 = \int_0^1 |v'|^2 \,\mathrm{d}x + |v(1)|^2.$$

Two equations

We insert $(\delta_\tau, 0)$ and obtain,

$$(1-\alpha)\left(\int_0^1 \tau'\,\delta_\tau'\,\mathrm{d}x + \tau(0)\,\delta_\tau\right) = \frac{1}{\epsilon}\int_0^1 \sigma\,\delta_\tau\,\mathrm{d}x + \int_0^1 u\,\delta_\tau'\,\mathrm{d}x.$$

Vice versa $(0, \delta_v)$

$$\alpha\left(\int_0^1 v'\,\delta_v'\,\mathrm{d}x + v(1)\delta_v(1)\right) = \int_0^1 \sigma\,\delta_v'\,\mathrm{d}x - \int_0^1 u\,\delta_v'\,\mathrm{d}x + \hat{\sigma}(0)\delta_v(0) - \hat{\sigma}(1)\delta_v$$

Reformulate

We have two equations for τ and v,

$$\begin{cases} -(1-\alpha)\tau'' = \dfrac{1}{\epsilon}\sigma - u' \\ (1-\alpha)\tau'(1) = u(1) \\ (1-\alpha)(-\tau'(0)+\tau(0)) = -u(0), \end{cases}$$

$$\begin{cases} -\alpha v'' = -\sigma' + u' \\ \alpha(v'(1) + v(1) = \sigma(1) - u(1)) - \hat{\sigma}(1) \\ -\alpha\, v'(0) = -\sigma(0) + u(0) + \hat{\sigma}(0) \end{cases}$$

Solving for τ and v yields

$$(1-\alpha)\tau' = \frac{1}{\epsilon}\int_x^1 \sigma \, \mathrm{d}s + u,$$
$$(1-\alpha)\tau(0) = \int_0^1 \sigma \, \mathrm{d}s,$$

and

$$\alpha v' = \sigma - u - \hat{\sigma}(0),$$
$$\alpha v(1) = \hat{\sigma}(0) - \hat{\sigma}(1),$$

respectively.

The final result

Explicit form

$$\|(\sigma,\hat\sigma(0),\hat\sigma(1),u)\|_E^2 := \|\frac{1}{\epsilon}\int_x^1 \sigma\,\mathrm{d}s + u\|^2 + |\frac{1}{\epsilon}\int_0^1 \sigma\,\mathrm{d}x|^2 + \|\sigma - u - \hat\sigma(0)\|^2 + |\hat\sigma(0)-\hat\sigma(1)|^2.$$

Theorem

There exists a constant C independent of ϵ, such that

$$\max\{\|\sigma\|, \epsilon^{1/2}\|u\|, \epsilon^{1/2}\|\frac{1}{\epsilon}\int_{\cdot}^1 \sigma(s)\mathrm{d}s\|, \epsilon^{1/2}|\hat\sigma(0)|, \epsilon^{1/2}|\hat\sigma(1)|\} \leq C\|(\sigma,\hat\sigma(0),\hat\sigma(1),u)\|_E.$$

Proof.

On the blackboard. □

1D - Multielement (Composite DPG)

The variational formulation

For each $k = 1, \ldots, N$, find $\sigma_k, u_k \in L_2(x_{k-1}, x_k)$, $\hat{\sigma}(x_k) \in \mathbb{R}$, $\hat{u}(x_k) \in \mathbb{R}$, with $\hat{u}(0) = u_0$ and $\hat{u}(1) = 0$ s.t.

$$\begin{aligned}
&\frac{1}{\epsilon}\int_{x_{k-1}}^{x_k} \sigma_k\,\tau\,\mathrm{d}x + \int_{x_{k-1}}^{x_k} u_k\,\tau'\,\mathrm{d}x \\
&\quad -\hat{u}(x_k)\,\tau(x_k) + \hat{u}(x_{k-1})\,\tau(x_{k-1}) = 0 \\
&\int_{x_{k-1}}^{x_k} \sigma_k\,v'\,\mathrm{d}x - \int_{x_{k-1}}^{x_k} u_k\,v'\,\mathrm{d}x \\
&\quad -\hat{\sigma}(x_k)v(x_k) + \hat{\sigma}(x_{k-1})v(x_{k-1}) \\
&\quad +\hat{u}(x_k)\,v(x_k) - \hat{u}(x_{k-1})\,v(x_{k-1}) = \int_{x_{k-1}}^{x_k} f\,v\,\mathrm{d}x
\end{aligned}$$

for all $(\tau, v) = (\tau_k, v_k) \in H^1(x_{k-1}, x_k)$.

The choice of $(\,.\,,\,.\,)_V$

We choose $(\,.\,,\,.\,)_V$ associated to the norm

$$\|(\boldsymbol{\tau},\boldsymbol{v})\|^2 := \sum_{k=1}^{N} \|\tau_k\|^2 + \|v_k\|^2,$$

with

$$\|\tau_k\|^2 = \int_{x_{k-1}}^{x_k} |\tau_k'|^2 \,\mathrm{d}x + |\tau_k(x_{k-1})|^2,$$
$$\|v_k\|^2 = \int_{x_{k-1}}^{x_k} |v_k'|^2 \,\mathrm{d}x + |v_k(x_k)|^2.$$

The local problems

We insert $(\delta_\tau, 0)$ and obtain,

$$\int_{x_{k-1}}^{x_k} \tau' \, \delta_\tau' \, \mathrm{d}x + \tau(x_{k-1}) \, \delta_\tau(x_{k-1}) \\ = \frac{1}{\epsilon} \int_{x_{k-1}}^{x_k} \sigma_k \, \delta_\tau \, \mathrm{d}x + \int_{x_{k-1}}^{x_k} u_k \, \delta_\tau' \, \mathrm{d}x - \hat{u}(x_k) \, \delta_\tau(x_k) + \hat{u}(x_{k-1}) \, \delta_\tau(x_{k-1})$$

Vice versa $(0, \delta_v)$

$$\int_{x_{k-1}}^{x_k} v' \, \delta_v' \, \mathrm{d}x + v(x_k)\delta_v(x_k) \\ = \int_{x_{k-1}}^{x_k} \sigma_k \, \delta_v' \, \mathrm{d}x - \int_{x_{k-1}}^{x_k} u_k \, \delta_v' \, \mathrm{d}x \\ + \hat{\sigma}(x_{k-1})\delta_v(x_{k-1}) - \hat{\sigma}(x_k)\delta_v(x_k) + \hat{u}(x_k) \, \delta_v(x_k) - \hat{u}(x_{k-1}) \, \delta_v(x_{k-1})$$

Local Reformulation

For each $k = 1, \dots, N$, we have two equations for τ and v,

$$\begin{cases} \tau'' = \dfrac{1}{\epsilon}\sigma_k - u_k' \\ \tau'(x_k) = u_k(x_k) - \hat{u}(x_k) \\ -\tau'(x_{k-1}) + \tau(x_{k-1}) = u_k(x_{k-1}) + \hat{u}(x_{k-1}) \end{cases}$$

$$\begin{cases} v'' = -\sigma_k' + u_k' \\ v'(x_k) + v(x_k) = \sigma_k(x_k) - u_k(x_k) - \hat{\sigma}(x_k) + \hat{u}(x_k) \\ v'(x_{k-1}) = -\sigma_k(x_{k-1}) + u_k(x_{k-1}) + \hat{\sigma}(x_{k-1}) - \hat{u}(x_{k-1}) \end{cases}$$

The optimal test functions I

For each $k = 1, \dots, N$, we obtain

$$\begin{aligned}
\tau'(x) =& \frac{1}{\epsilon} \int_x^{x_k} \sigma_k(s)\,\mathrm{d}s + u_k(x) - \hat{u}(x_k) \\
\tau(x_{k-1}) =& \frac{1}{\epsilon} \int_{x_{k-1}}^{x_k} \sigma_k(s)\,\mathrm{d}x + \hat{u}(x_{k-1}) - \hat{u}(x_k) \\
\tau(x) =& \int_{x_{k-1}}^{x} (s - x_{k-1})\sigma_k(s)\,\mathrm{d}s + (x - x_{k-1}) \int_x^{x_k} \sigma_k(s)\,\mathrm{d}s \\
&+ \int_{x_{k-1}}^{x} u_k(s)\,\mathrm{d}s + \frac{1}{\epsilon} \int_{x_{k-1}}^{x_k} \sigma_k(s)\,\mathrm{d}x + \hat{u}(x_{k-1}) - \hat{u}(x_k)(x - x_{k-1} + 1)
\end{aligned}$$

The optimal test functions II

and

$$\begin{aligned}
v'(x) =& \sigma_k(x) - u_k(x) - \hat{\sigma}(x_{k-1}) + \hat{u}(x_{k-1}) \\
v(x_k) =& \hat{\sigma}(x_{k-1}) - \hat{\sigma}(x_k) - \hat{u}(x_{k-1}) + \hat{x_k} \\
v(x) =& \int_{x_{k-1}}^{x} \sigma_k(s)\ \mathrm{ds} + \int_{x_{k-1}}^{x} u_k(s)\ \mathrm{ds} + \hat{\sigma}(x_{k-1})(1 - x + x_{k-1}) \\
& + \hat{u}(x_{k-1})(x - x_{k-1} - 1) - \hat{\sigma}(x_k) + \hat{u}(x_k).
\end{aligned}$$

The optimal test functions III

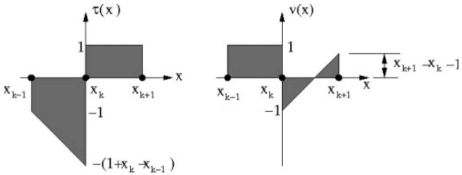


Figure: Optimal test functions for the unit flux $\hat{u}(x_k) = 1$.

No inf-sup condition is shown!

Numerical considerations

Assume $\sigma_k, u_k \in \mathcal{P}^p(x_{k-1}, x_k)$, then

$$\tau = \tau_k \in \mathcal{P}^{p+2}(x_{k-1}, x_k) \quad \text{and} \quad v = v_k \in \mathcal{P}^{p+1}(x_{k-1}, x_k).$$

When we choose the test functions as in the pure convection problem, i.e. higher order polynomials, we would obtain

$$\begin{matrix} 2N(p+1) \text{ unknowns for } \sigma_k, u_k \\ +\, 2N \text{ unknowns for the fluxes} \end{matrix} \to \mathbf{N(2p+4)} \text{ unknowns}$$

but

$$N(p+2+p+3) \text{ unknowns for } \tau, v \to \mathbf{N(2p+5)} \text{ unknowns.}$$

Numerical Experiment

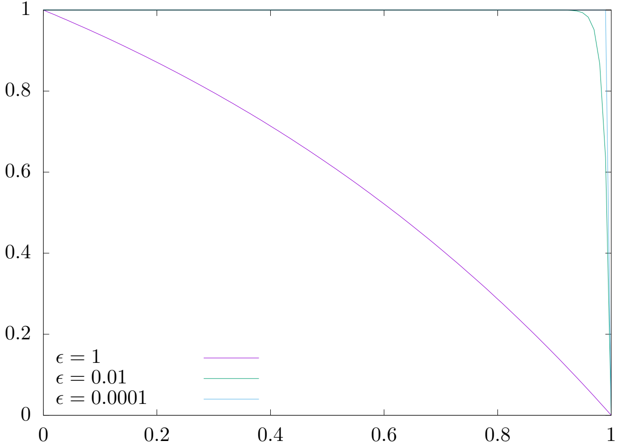
Implementation details

- $\sigma_k, u_k \in \mathcal{P}^p(x_{k-1}, x_k)$
- standard H^1-norm for the computation of the optimal test functions
- $\tau_k, v_k \in \mathcal{P}^{p+3}(x_{k-1}, x_k)$
- rhs $f \equiv 0$, $u_0 = 1$, solution is

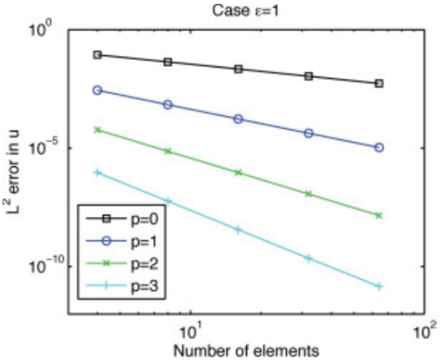
$$\sigma(x) = -\frac{1}{1 - e^{-\frac{1}{\epsilon}}} e^{\frac{x-1}{\epsilon}}$$

$$u(x) = \frac{1}{1 - e^{-\frac{1}{\epsilon}}} \left(1 - e^{\frac{x-1}{\epsilon}}\right)$$

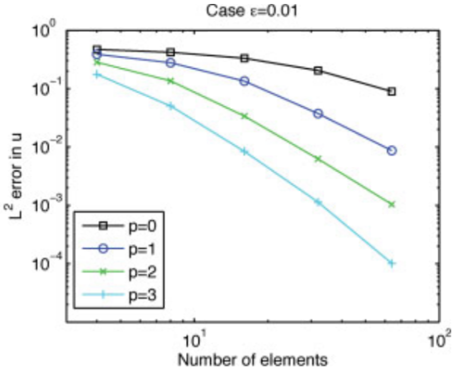
Plot of the analytical solution



Numerical Results I

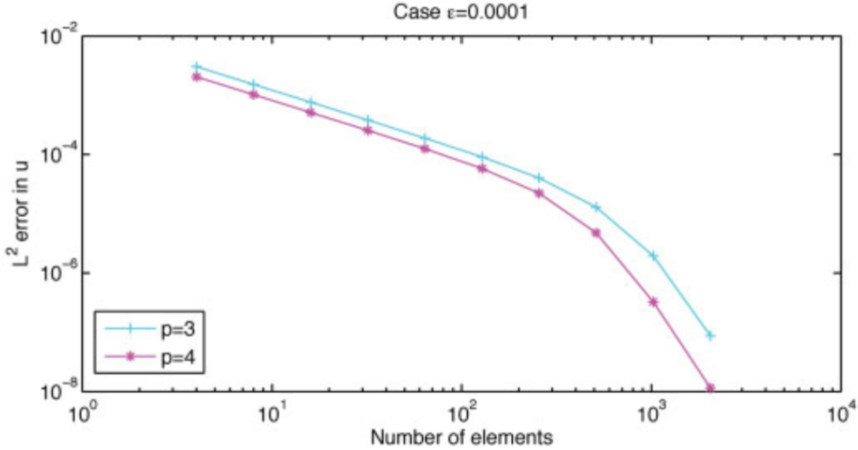


(a) Case $\epsilon = 1$, $p = 0, 1, 2, 3$.



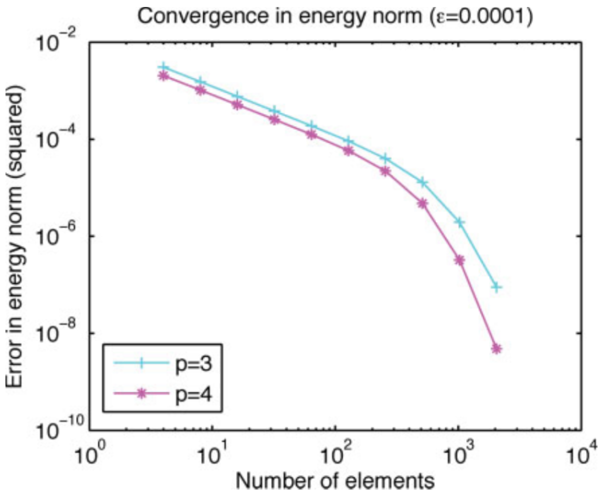
(b) Case $\epsilon = 0.01$, $p = 0, 1, 2, 3$.

Numerical Results II



(c) Case $\epsilon = 10^{-4}$, $p = 3, 4$.

Numerical Results III



A 2D model problem

Classical (mixed) form

Find u, σ, s.t.,

$$\begin{aligned} \frac{1}{\varepsilon}\boldsymbol{\sigma} - \nabla u &= 0 && \text{in } \Omega, \\ -\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{\beta}\cdot\nabla u &= f && \text{in } \Omega, \\ u &= u_0 && \text{on } \partial\Omega. \end{aligned}$$

A 2D model problem

Variational Form

For each element K in the mesh, find $\boldsymbol{\sigma}_K \in \boldsymbol{L}_2(K)$, $u_K \in L_2(K)$ and fluxes $\hat{u}_e \in U(e), \hat{\sigma}_e \in L_2(e)$, s.t.,

$$\frac{1}{\epsilon}\int_K \boldsymbol{\sigma}_K \,\boldsymbol{\tau}\,\mathrm{d}x + \int_K u_K \,\operatorname{div}\boldsymbol{\tau}\,\mathrm{d}x - \sum_{e\in\partial K\setminus\partial\Omega}\int_e \tau_n\,\hat{u}_e\,\mathrm{d}s_x = l_1(\boldsymbol{\tau}),$$

$$\int_K \boldsymbol{\sigma}_K\cdot\nabla v\,\mathrm{d}x - \int_K u_K\,\boldsymbol{\beta}\cdot\nabla v\,\mathrm{d}x$$

$$+ \sum_{e\in\partial K\setminus\partial\Omega}\int_e \beta_n\,\hat{u}_e v\,\mathrm{d}s_x - \sum_{e\in\partial K}\int_e \hat{\sigma}_e\,\operatorname{sgn}(\boldsymbol{n}_K)\,v\,\mathrm{d}s_x = l_2(v)$$

for all $\boldsymbol{\tau} \in H(\operatorname{div}, K)$ and $v \in H^1(K)$.

Some remarks I

Sign of the flux

Each edge e has normal $\boldsymbol{n}_e$, then

$$\operatorname{sgn}(\boldsymbol{n}_K) := \begin{cases} 1, & \text{if } \boldsymbol{n}_K = \boldsymbol{n}_e \\ -1, & \text{if } \boldsymbol{n}_K = -\boldsymbol{n}_e. \end{cases}$$

The space $U_e(K)$

- $\boldsymbol{\tau} \in H(\mathrm{div}, K) \Rightarrow \tau_n \in H^{-1/2}(\partial K)$
- $\hat{u}_e \in L_2(e)$ is not enough
- we **assume** well posedness (no theory provided)

Some remarks II

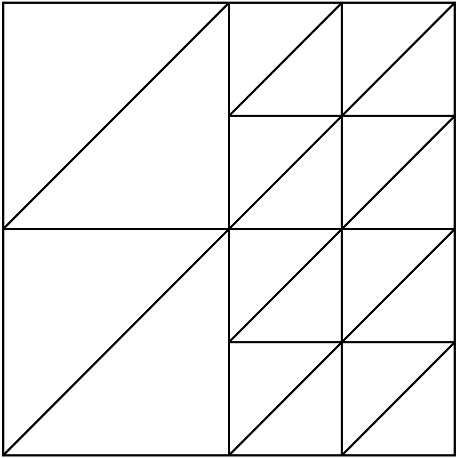


Figure: We allow one hanging node per element.

The optimal test functions

Variational form

We choose the *standard scalar products* associated to $H(\mathrm{div},\Omega)$ and $H^1(\Omega)$. Then we solve: find $\boldsymbol{\tau} \in H(\mathrm{div},\Omega)$ and $v \in H^1(\Omega)$, s.t.

$$\int_K \mathrm{div}\,\boldsymbol{\tau}\,\mathrm{div}\,\boldsymbol{\delta_\tau} + \boldsymbol{\tau}\,\boldsymbol{\delta_\tau}\;\mathrm{d}x = \frac{1}{\epsilon}\int_K \boldsymbol{\sigma}_K \boldsymbol{\delta_\tau}\;\mathrm{d}x + \int_K u_K\;\mathrm{div}\,\boldsymbol{\delta_\tau}\;\mathrm{d}x - \sum_{e\in\partial K\setminus\partial\Omega}\int_e \delta_{\tau\,n}\hat{u}_e\;\mathrm{d}s_x$$

for all $\boldsymbol{\delta_\tau} \in H(\mathrm{div},\Omega)$ and

$$\int_K \nabla v\cdot\nabla\delta_v + v\,\delta_v\;\mathrm{d}x = \int_K \boldsymbol{\sigma}_K \nabla\delta_v\;\mathrm{d}x - \int_K u_K \boldsymbol{\beta}\cdot\nabla\delta_v\;\mathrm{d}x + \sum_{e\in\partial K\setminus\partial\Omega}\int_e \beta_n\hat{u}_e\,\delta_v\;\mathrm{d}s_x - \sum_{e\in\partial K}\int_e \hat{\sigma}_e\,\mathrm{sgn}(\boldsymbol{n}_K)\delta_v\;\mathrm{d}s_x$$

for all $\delta_v \in H^1(\Omega)$.

Numerical considerations & details

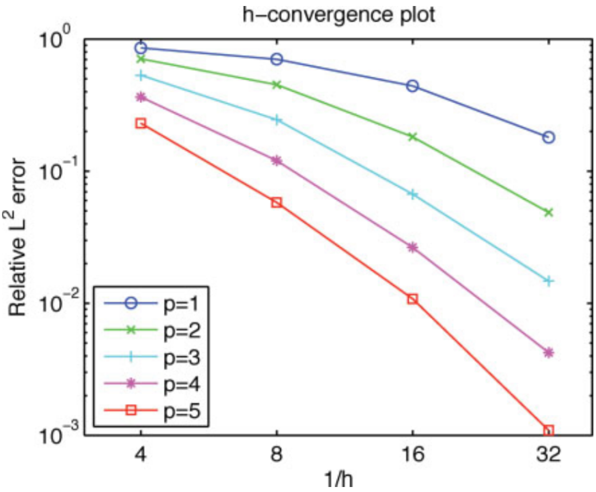
- triangular meshes with at most one hanging node per element K
- $\boldsymbol{\sigma}_K, u_k \in \mathcal{P}^p(K)$
- order of flux over an edge e is maximum order of adjacent elements
- for edges with hanging nodes, piecewise polynomials are used
- optimal test functions are approximated in $\mathcal{P}^{p+3}(K)$

Numerical experiment

- $\Omega = (0,1)^2$
- $\epsilon = 0.01$, $\boldsymbol{\beta} = (2,1)^T$
- solution

$$u(x,y) = \left(x + \frac{e^{\frac{\beta_1 x}{\epsilon}} - 1}{1 - e^{\frac{\beta_1}{\epsilon}}}\right)\left(y + \frac{e^{\frac{\beta_2 y}{\epsilon}} - 1}{1 - e^{\frac{\beta_2}{\epsilon}}}\right)$$

Numerical Results



Conclusions

- Boundary value problem is reformulated as a system of first order PDEs
- mesh dependent (ultra-)weak formulation
- Variational equations for each element, connected via fluxes
- Exact optimal test functions give best approximation error
- Method gives L_2-stability regardless of ϵ
- Resulting global stiffness matrix is SPD