DPG method with optimal test functions for thin-body problems in solid mechanics

Ludwig Mitter

JOHANNES KEPLER UNIVERSITY LINZ

ludwig.mitter@numa.uni-linz.ac.at

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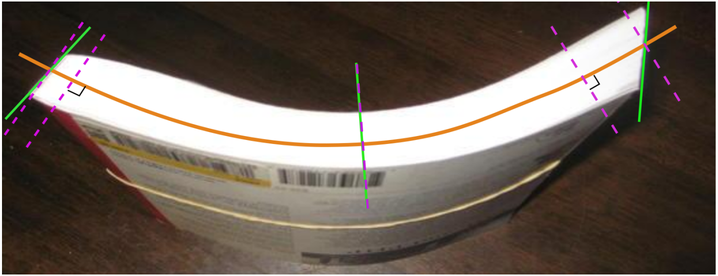
Overview

Dimensional Reduction in Continuum Mechanics

Why dimensional reduction for thin bodies?

1. Avoid **geometry locking** with standard discretization
2. Avoid very **complex three-dimensional discretization** that is stable wrt. thickness
3. Replace three-dimensional problem with **two- or one-dimensional problem**



Dimensional Reduction in Continuum Mechanics

How does it work?

1. Claim that three-dimensional displacement has specific form (**lower-dimensional parameter functions**)
2. Insert ansatz into three-dimensional model and **perform calculations analytically** as much as possible

What do we get?

BEAMS	Straight 1D bodies
ARCHES	Curved 1D bodies
PLATES	Planar 2D bodies
SHELLS	Curved 2D bodies

The Timoshenko beam model

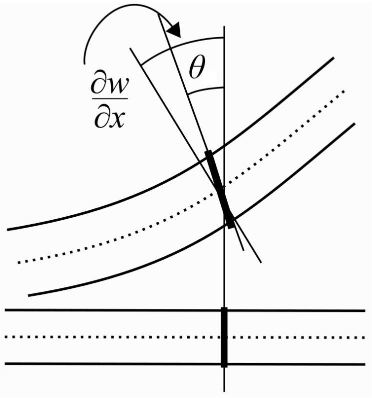
Displacement

$$\left[u_1, u_2, u_3\right](x_1, x_2, x_3) =$$
$$\left[-x_3\theta(x_1), 0, w(x_1)\right]^\top$$

w...transverse deflection
θ...rotation

Perpendicular loading

$$\left[f_1, f_2, f_3\right](x_1, x_2, x_3) =$$
$$\left[0, 0, p(x_1)\right]^\top$$



Derivation of the Timoschenko system

$$\boldsymbol{E} = \begin{bmatrix} -x_3\theta'(x_1) & 0 & \frac{1}{2}(w'(x_1) - \theta(x_1)) \\ 0 & 0 & 0 \\ \frac{1}{2}(w'(x_1) - \theta(x_1)) & 0 & 0 \end{bmatrix}$$

Modified constitutive law for **consistency**.

$$\boldsymbol{\Sigma} = \begin{bmatrix} -Ex_3\theta'(x_1) & 0 & \kappa\mu(w'(x_1) - \theta(x_1)) \\ 0 & -E\nu x_3\theta'(x_1) & 0 \\ \kappa\mu(w'(x_1) - \theta(x_1)) & 0 & 0 \end{bmatrix}$$

Assume $\Omega = (0, L) \times \omega$. Set $I := \int_\omega x_3^2 dx_3, A := |\omega|$.

$$\Big(\boldsymbol{\Sigma}, \boldsymbol{E}\Big)_{0,\Omega} = EI \int_a^b \theta'(x1)^2 dx_1 + \kappa\mu A \int_a^b \Big(w'(x_1) - \theta(x_1)\Big)^2 dx_1$$

The Timoschenko system

Find w, θ such that

$$-\frac{\partial}{\partial x_1}\left[EI\frac{\partial \theta}{\partial x_1}\right] + \kappa\mu A\left(\frac{\partial w}{\partial x_1} - \theta\right) = m \text{ in } (0, L)$$

$$-\frac{\partial}{\partial x_1}\left[\kappa\mu A\left(\frac{\partial w}{\partial x_1} - \theta\right)\right] = p \text{ in } (0, L)$$

Introducing auxiliary variables and rescaling

1. **Auxiliary variables:**

$$V = \kappa\mu A(w' - \theta) \qquad M = EI\theta'$$

leads to

$$-V' = p \qquad -M' - V = m$$

2. **Rescaling:**

$$x_1 \hookrightarrow Lx_1 \quad w \hookrightarrow Lw \quad \theta \hookrightarrow \theta$$
$$V \hookrightarrow \mu AV \quad M \hookrightarrow EIL^{-1}M \quad p \hookrightarrow \mu AL^{-1}p \quad m \hookrightarrow EIL^{-2}m$$

leads to **dimensionless system**

$$V = \kappa(w' - \theta) \qquad M = \theta' \text{ in } (0, L)$$
$$-V' = p \qquad -M' - kV = m \text{ in } (0, L)$$

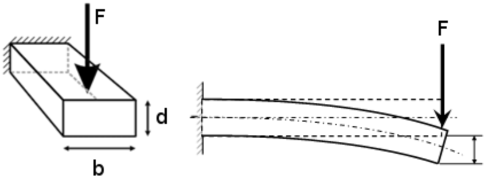
where $k = \frac{\mu AL^2}{EI}$

Assumptions

1. Crossection of beam is **rectangle** (width b, thickness d)
   $\implies A = bd, I = \frac{bd^3}{12}, k = \frac{12\mu\epsilon^2}{E}$
   where $\epsilon = \frac{d}{L}$ **dimensionless thickness**
2. Assume **thin beam** $\epsilon \ll 1$
   $\implies$ **rescaling** $V \hookrightarrow \epsilon^{-2} V, p \hookrightarrow \epsilon^{-2} p$ which leads to

$$\epsilon^{-2} V = \kappa(w' - \theta) \qquad M = \theta' \text{ in } (0,1)$$
$$-V' = p \qquad -M' - kV = m \text{ in } (0,1)$$

   where $k = \frac{12\mu}{E}$



The Model Problem

1. **Volume loads:** $p \equiv 0, m \equiv 0$
2. **Surface load** at $\{L\} \times \omega$ of magnitude $(0, 0, F)^\top$
3. **Clamped** at $\{0\} \times \omega$

This leads to the

MODEL PROBLEM

Find V, M, w, θ such that

$$\epsilon^{-2} V = \kappa(w' - \theta) \qquad M = \theta' \text{ in } (0,1)$$

$$-V' = p \qquad -M' - kV = m \text{ in } (0,1)$$

$$V(1) = F \qquad M(1) = 0 \qquad w(0) = 0 \qquad \theta(0) = 0$$

where $k = \frac{12\mu}{E}$

Classical solution

This problem can be solved explicitly:

$$V = F \qquad M = -kF(x_1 - 1)$$

$$\theta = kF\left(x_1 - \frac{1}{2}x_1^2\right) \qquad w = \frac{kF}{2}\left(x_1^2 - \frac{x_1^3}{3}\right) + \frac{F}{\epsilon^2\kappa}x_1$$

The weak ONE-ELEMENT formulation

Find $\boldsymbol{u} = (V, M, w, \theta) \times (\hat{V}(0), \hat{M}(0), \hat{w}(1), \hat{\theta}(1)) \in \mathcal{U}$ such that

$$
\begin{aligned}
\epsilon^2(V, q) + \kappa(w, q') - \kappa\hat{w}(1)q(1) + \kappa(\theta, q) &= 0 \\
(M, \tau) + (\theta, \tau') - \hat{\theta}(1)\tau(1) &= 0 \\
(V, z') + \hat{V}(0)z(0) &= Fz(1) \\
k(V, \phi) - (M, \phi') - \hat{M}(0)\phi(0) &= 0
\end{aligned}
$$

for all $\boldsymbol{v} = (q, \tau, z, \phi) \in \mathcal{V}$, where $(a, b) := \int_0^1 abdx_1$ and $\mathcal{U} := [L^2(0,1)]^4 \times \mathbb{R}^4, \mathcal{V} = [H^1(0,1)]^4$.

Short form:

$$\text{Find } \boldsymbol{u} \in \mathcal{U} : B(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v}), \forall \boldsymbol{v} \in \mathcal{V}$$

Well-posedness

1. Equip Hilbert space $\mathcal{U}$ with *energy norm*

$$|||\boldsymbol{u}||| := \sup_{\boldsymbol{v}\in\mathcal{V}} \frac{B(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{\mathcal{V}}}$$

2. Regular Hilbert space norm $\|.\|_{\mathcal{V}}$ corresponding to $(.,.)_{\mathcal{V}}$

$$\|\boldsymbol{v}\|_{\mathcal{V}}{}^2 = \|q\|_V^2 + \|\tau\|_V^2 + \|z\|_V^2 + \|\phi\|_V^2 \quad \|v\|_V^2 = \|v\|^2 + \|v'\|^2$$

3. **Reminder:** Then one has $|||\boldsymbol{u}||| = \|T\boldsymbol{u}\|_{\mathcal{V}}$ with $T\boldsymbol{u}$ as

$$(T\boldsymbol{u},\boldsymbol{v})_{\mathcal{V}} = B(\boldsymbol{u},\boldsymbol{v}), \forall \boldsymbol{v} \in \mathcal{V}$$

4. Equip Hilbert space $\mathcal{U}$ with *"more" standard norm*

$$\|\boldsymbol{u}\|_{\mathcal{U}} := \max\{\|V\|, \|M\|, \|w\|, \|\theta\|, \left|\hat{V}(0)\right|, \left|\hat{M}(0)\right|, |\hat{w}(1)|, \left|\theta(\hat{1})\right|\}$$

Well-posedness

Theorem

Let $\boldsymbol{u} \in \mathcal{U}$. Then there eist two consts. $c_1, c_2 > 0$ independent of ϵ such that

$$c_1 \|\boldsymbol{u}\|_{\mathcal{U}} \leq \|\boldsymbol{u}\| \leq c_2 \|\boldsymbol{u}\|_{\mathcal{U}}.$$

Hence, $B(.,.)$ is bounded and satisfies inf-sup.

Proof.

Construction of an explicit expression for the energy norm AND replacement of $\|.\|_V = \|.\|_{H^1}$ above with

$$\|v\|_V^2 := \left\|v'\right\|^2 + |v(1)|^2$$

Blackboard. □

PG with optimal test functions

1. Let $\mathcal{U}_n = span\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\} \subset \mathcal{U}$
2. Optimal test space $\mathcal{V}_n^{\text{opt}} = span\{T\boldsymbol{e}_1, \cdots, T\boldsymbol{e}_n\}$
3. **Symmetrized PG method** (energy projection)

   $$\text{Find } \boldsymbol{u}_n \in \mathcal{U}_n : B(\boldsymbol{u}_n, \boldsymbol{v}_n) = L(\boldsymbol{v}_n), \forall \boldsymbol{v}_n \in \mathcal{V}_n^{\text{opt}}.$$

   Reminder:

   $\boldsymbol{v}_n = T\boldsymbol{w}_n, \boldsymbol{w}_n \in \mathcal{U}_n \implies (T\boldsymbol{u}_n, T\boldsymbol{w}_n) = L(T\boldsymbol{w}_n)$
4. **Cea:**

   $$\|\boldsymbol{u} - \boldsymbol{u}_n\|_{\mathcal{U}} \leq \frac{c_1}{c_2} \min_{\boldsymbol{w}_n \in \mathcal{U}_n} \|\boldsymbol{u} - \boldsymbol{w}_n\|_{\mathcal{U}}$$

DPG with optimal test functions

Now the **ULTRA-weak formulation**

1. Additionally to
$\boldsymbol{u}_n = (V, M, w, \theta) \times (\hat{V}(0), \hat{M}(0), \hat{w}(1), \hat{\theta}(1)) \in \mathcal{U}_n$
include **interface variables**
$\boldsymbol{\lambda}_n = (\hat{\boldsymbol{V}}, \hat{\boldsymbol{M}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\theta}}) \in \Lambda_n = \mathbb{R}^{4N-4}$ corresponding to

$$\mathcal{T}_n : 0 = x_0 < x_1 < \cdots < x_N = 1$$

2. **DPG bilinear form** where
$\boldsymbol{v}\big|_K = (q_j, \tau_j, z_j \phi_j) \in \mathcal{V}(K) = [H^1(K)]^4$:

$$B_h(\boldsymbol{u}_n, \boldsymbol{\lambda}_n; \boldsymbol{v}) = \sum_{j=1}^{N} \Big[\epsilon^2 \int_{x_{j-1}}^{x_j} V q_j dx + \kappa \int_{x_{j-1}}^{x_j} w q_j' dx$$

$$- \kappa \hat{w}(x) q_j(x)\big|_{x_{j-1}}^{x_j} + \kappa \int_{x_{j-1}}^{x_j} \theta q_j dx + \int_{x_{j-1}}^{x_j} M \tau_j dx + \int_{x_{j-1}}^{x_j} \theta \tau_j' dx$$

$$- \hat{\theta}(x) \tau_j(x)\big|_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j} V z_j' dx - \hat{V}(x) z_j(x)\big|_{x_{j-1}}^{x_j} + k \int_{x_{j-1}}^{x_j} V \phi_j dx$$

$$- \int_{x_{j-1}}^{x_j} M \phi_j' dx + \hat{M}(x) \phi_j(x)\big|_{x_{j-1}}^{x_j} \Big]$$

DPG with optimal test functions

1. **Discont. test space**: $\|\boldsymbol{v}\|_{\mathcal{V}_n} := \sum_K \|\boldsymbol{v}\|_{\mathcal{V}(K)}$

$$\mathcal{V}_n := \{\boldsymbol{v} \in [L^2(0,1)]^4, v\big|_K \in \mathcal{V}(K), K \in \mathcal{T}_n\}$$

2. **New optimal test space:**
$\mathcal{W}_n^{\text{opt}} = span\{T_n\boldsymbol{e}_1, \cdots, T_n\boldsymbol{e}_n, T_n\boldsymbol{\lambda}_n\}$ where
$T_n : \mathcal{U}_n \times \Lambda_n \to \mathcal{V}_n$ such that

$$(T_n(\boldsymbol{u}_n, \boldsymbol{\lambda}_n), \boldsymbol{v})_{\mathcal{V}_n} = B_n(\boldsymbol{u}_n, \boldsymbol{\lambda}_n; \boldsymbol{v}), \forall \boldsymbol{v} \in \mathcal{V}_n$$

DPG problem

Find $\boldsymbol{u}_n \in \mathcal{U}_n, \boldsymbol{\lambda}_n \in \Lambda_n$ such that

$$B_n(\boldsymbol{u}_n, \boldsymbol{\lambda}_n; \boldsymbol{v}) = L_n(\boldsymbol{v}), \forall \boldsymbol{v} \in \mathcal{W}_n^{\text{opt}}$$

The LOCALIZATION PRINCIPLE

Reminder

1. Element-wise computation of $\mathcal{W}_n^{\mathrm{opt}}$
2. $B_n(\boldsymbol{u}_n, \boldsymbol{\lambda}_n; \boldsymbol{v}) = B(\boldsymbol{u}_n, \boldsymbol{v})$ and $\|\boldsymbol{v}\|_{\mathcal{V}_n} = \|\boldsymbol{v}\|_{\mathcal{V}}$ for $\boldsymbol{v} \in \mathcal{V}$

Lemma (Localization principle)

One has

$$\mathcal{V}_n^{opt} \subset \mathcal{W}_n^{opt},$$

consequently **ONE-ELEMENT and DPG coincide**.

Theorem (Best-approx. property)

1. $\boldsymbol{u} = (V, M, w, \theta)$ **one-element weak solution**
2. $\boldsymbol{u}_n = (V_n, M_n, w_n, \theta_n)$ **DPG solution**

Then $\boldsymbol{u}_n$ is the **best-approx.** *of $\boldsymbol{u}$ wrt. $\|.\|_{L^2}$ up to ϵ-indep. constant.*

Numerics

Remarks:

1. Optimal test fncts. corresponding to polynomial trial fncts. are NOT polynomials. Hence, resolve opt. test fncts. in **enriched FE space** (pol. degree $+1$)
2. Theoretical analysis of the **use of approx. opt. test fncts. is open** at the moment.
3. Numerics only for *Euler-Bernoulli* beam at $\epsilon = 0$ and $\epsilon = 0.1, \kappa = 5/6$ with **pw. const** and **pw. linear** elements.

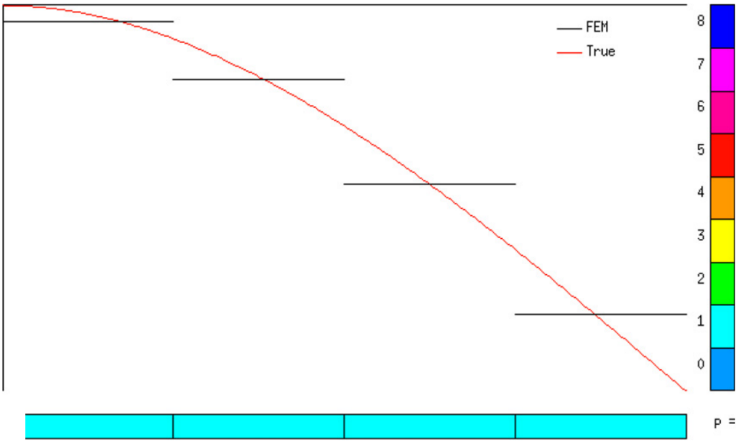
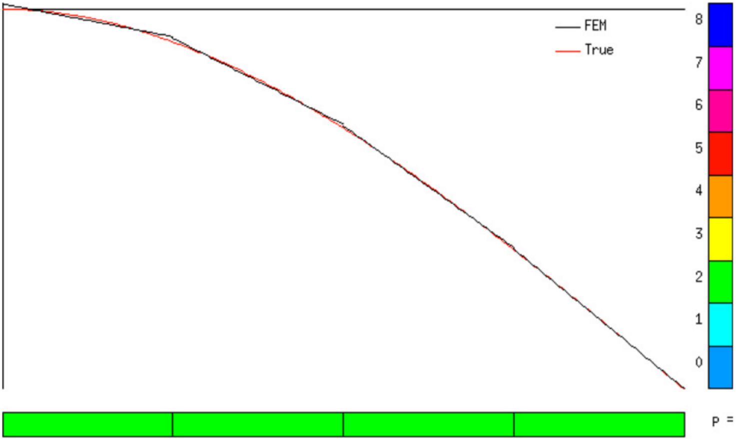


Figure: Transverse deflection w at $\epsilon = 0$.



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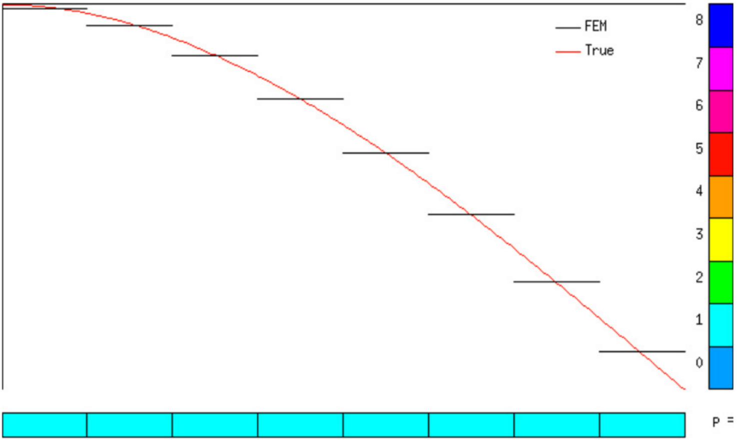

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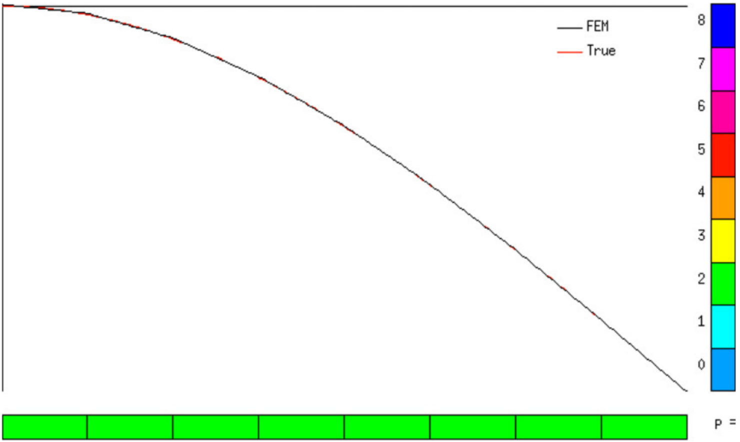

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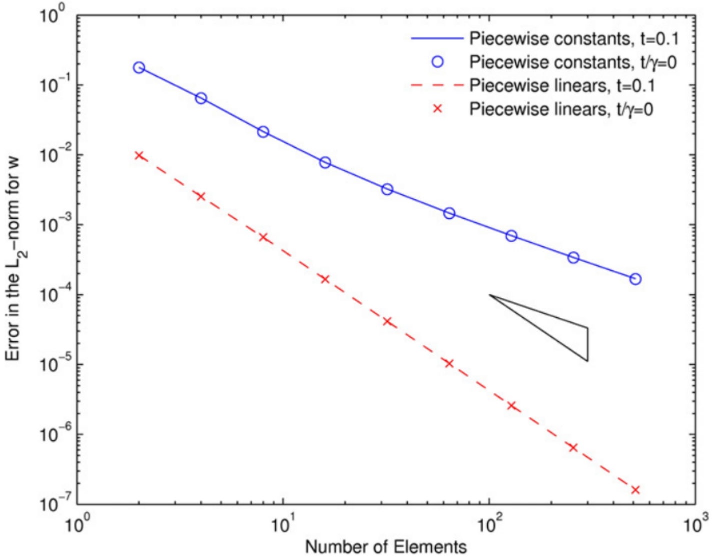


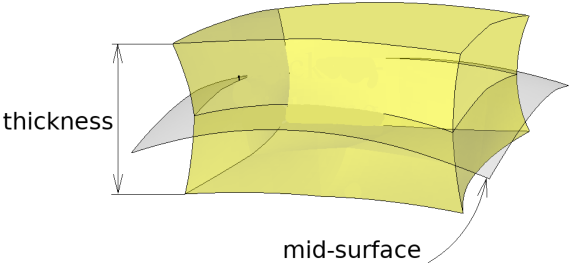
Figure: Convergence history for pw. constant and linear elements.

Basic edge effect in shell deformation: ASSUMPTIONS

NOW: Shells (RN shell model). Assume

1. *Axially symmetric* deformations of a *shallow spherical shell* with radius R and thickness d, $d \ll R$.
2. If x_1, x_2 mid-surface coords, NO deformation along x_2.
3. Homogeneous isotropic material.

THEN kinematics of shell given via single tangential displacement $u(x_1)$, transverse deflection $w(x_1)$ and single rotation of normal $\theta(x_1)$.



Governing equations

If one takes R as *length unit*, EdR as *force unit*, dimless. thickness $\epsilon = d/R$ one ends up with
Static equilibrium for Stresses T, Shear reaction V_x, Bending reaction M_x:

$$T_x = u' + w \qquad T_y = w$$

$$V_x = \kappa(w' - \theta) \qquad M_x = \frac{\epsilon^2}{12}\theta'$$

$$-T_x' = 0 \qquad T_x + T_y - V_x' = 0 \qquad -M_x' - V_x = 0$$

The model problem

1. **Assume:** $T_x = 0$
2. **Rescaling:** $V_x \hookrightarrow \epsilon^2 V, M_x \hookrightarrow \epsilon^2 k^{-1} M, k = 12$

MODEL PROBLEM

Find V, M, w, θ such that

$$\epsilon^2 V = \kappa(w' - \theta) \qquad M = \theta'$$

$$w - \epsilon^2 V' = 0 \qquad -M' - kV = 0$$

$$M(0) = 1 \qquad V(0), w(1), \theta(1) = \cdots$$

where $k = 12$

Classical solution

Cannot be resolved by direct elimination of the unknowns, however one can show

$$\boldsymbol{u}(x_1) = e^{\lambda x_1}\boldsymbol{U}$$
$$\implies \boldsymbol{u}(x_1) = e^{a x_1}\Big(\boldsymbol{A}(\epsilon)\cos(b x_1) + \boldsymbol{B}(\epsilon)\sin(b x_1)\Big)$$

where $a \approx b \approx \pm 3^{1/4}/\sqrt{\epsilon}$

The weak ONE-ELEMENT formulation

Similar setting as for Timoschenko-beam model:

$$\begin{aligned}
\boldsymbol{u} &= (V, M, \theta, w) \times (\hat{w}(0), \hat{\theta}(0), \epsilon^2 \hat{V}(1), \hat{M}(1)) \in \mathcal{U} \\
\boldsymbol{v} &= (q, \tau, z, \phi) \in \mathcal{V} \\
\mathcal{U} &= [L^2(0,1)]^4 \times \mathbb{R}^4 \qquad \mathcal{V} = [H^1(0,1)]^4
\end{aligned}$$

PG formulation

$$\text{Find } \boldsymbol{u} \in \mathcal{U} : B(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v}), \forall \boldsymbol{v} \in \mathcal{V},$$

where

$$\begin{aligned}
B(\boldsymbol{u}, \boldsymbol{v}) = \epsilon^2 (V, q) + \kappa (w, q') + \kappa \hat{w}(0) q(0) + \kappa (\theta, q) \\
+ (M, \tau) + (\theta, \tau') + \hat{\theta}(0) \tau(0) + (w, z) \\
+ \epsilon^2 (V, z') - \epsilon^2 \hat{V}(1) z(1) \\
+ k(V, \phi) - (M, \phi') + \hat{M}(1) \phi(1)
\end{aligned}$$

Choosing proper norms for ϵ-robustness

1. Timoschenko-$\|.\|_{\mathcal{V}}$ DOES NOT lead to equivalence of $|||.|||$ to $\|.\|_{L^2}$ uniformly in ϵ.
2. **INSTEAD:** Arrive at $|||.||| = \|.\|_{\mathcal{U}}$ by setting

$$\|\boldsymbol{v}\|_{\mathcal{V}} := \sup_{\boldsymbol{v}\in\mathcal{V}} \frac{B(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{u}\|_{\mathcal{U}}}$$

3. For our computations observe that

$$B(\boldsymbol{u},\boldsymbol{v}) = \left(V, \epsilon^2 q + \epsilon^2 z' + k\phi\right) + \left(M, \tau - \phi'\right) + \left(w, \kappa q' + z\right) + \left(\theta, \kappa q + \tau'\right) + \kappa\hat{w}(0)q(0)$$
$$\hat{\theta}(0)\tau(0) - \epsilon^2\hat{V}(1)z(1) + \hat{M}(1)\phi(1)$$

hence by using $\|.\|_{\mathcal{U}} = \max\{\cdots\}$ one has (1-elem. case)

$$\|\boldsymbol{v}\|_{\mathcal{V}}^2 = \left\|\epsilon^2(q+z') + k\phi\right\|^2 + \left\|\tau - \phi'\right\|^2 + \left\|\kappa q' + z\right\|^2 + \left\|\kappa q + \tau'\right\|^2 + |\kappa q(0)|^2 + |\tau(0)|^2 + |z(1)|^2 + |\phi(1)|^2$$

Choosing proper norms for ϵ-robustness

1. Replace with **localizable norm**

$$\|\boldsymbol{v}\|_{\mathcal{V}}{}^2 = \left\|\epsilon^2(q+z') + k\phi\right\|^2 + \left\|\tau - \phi'\right\|^2 + \left\|\kappa q' + z\right\|^2 + \left\|\kappa q + \tau'\right\|^2 + \|\kappa q\|^2 + \|\tau\|^2 + \|z\|^2 + \|\phi\|^2$$

2. **Question remains open:** Are these two norms uniformly equivalent wrt. ϵ for DPG setting? (Numerics indicate robustness)
3. **OBSERVE:** Using above **optimal test space norm** instead of **standard test space norm** (as for Timoschenko), boundary layer effects have to be resolved by test fncts.! Hence, higher enrichment degree for FEM approx. of test fncts.

Numerics

Remarks:

1. Trial fncts. are pw. pols. of equal order.
2. Optimal test space degree enrichment (+3).
3. Use both: **Optimal/Standard test space norms**.
4. Adaptive refinement (Endtmayer)

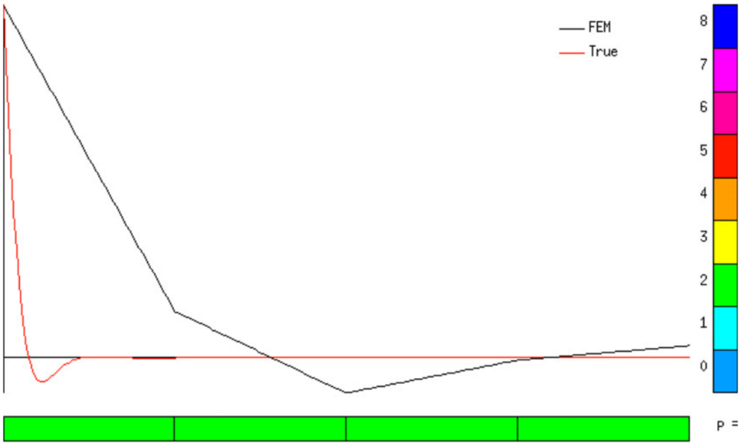


Figure: Bending moment M at $\epsilon = 1.e - 3$. OPTIMAL test space norm.

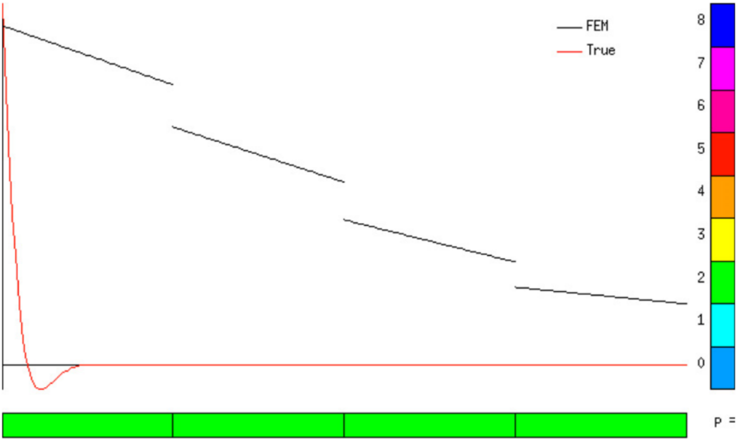
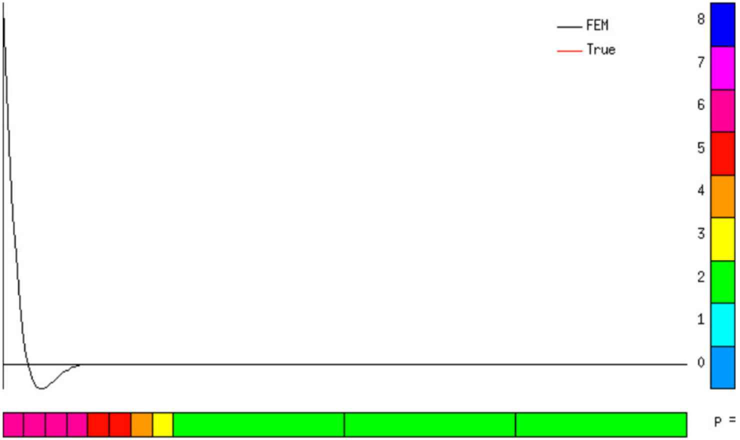


Figure: Bending moment M at $\epsilon = 1.e - 3$. STANDARD test space norm.



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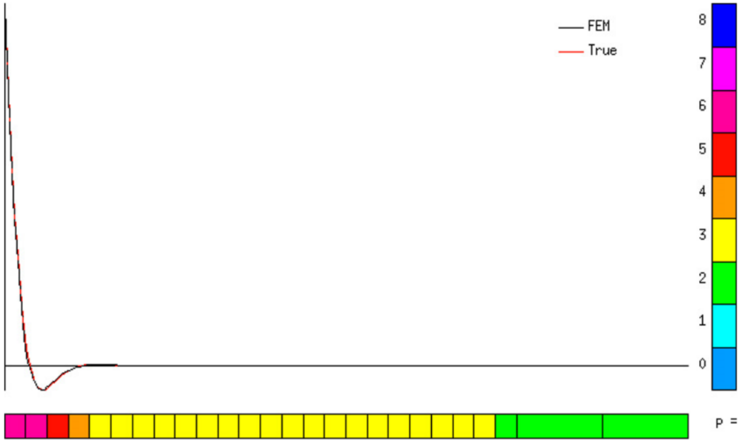

Figure: Bending moment M at $\epsilon = 1.e - 3$. STANDARD test space norm.

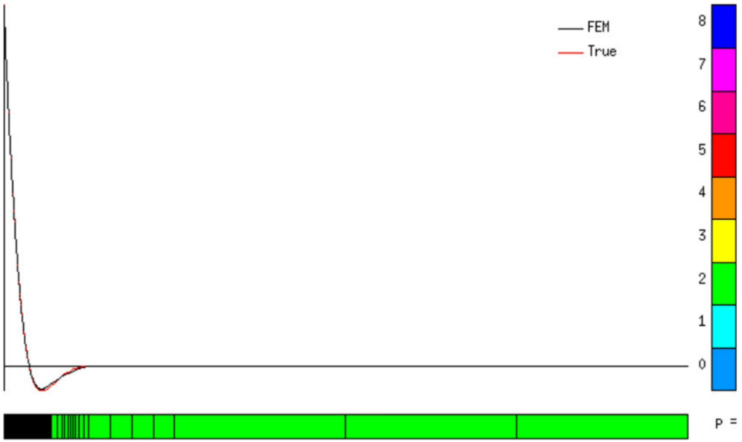

Figure: Bending moment M at $\epsilon = 1.e-3$. OPTIMAL test space norm.

Figure: Bending moment M at $\epsilon = 1.e - 3$. OPTIMAL test space norm.

Why **could** DPG methods be interesting for shell simulations?

1. Various **locking** effects disturb **FEM modelling of shells**.
2. **Beams**, Arches, Plates have a **relatively simple asymptotic behaviour** for $\epsilon \to 0$
3. **SHELL-asymptotics difficult**: Breaks into several subproblems (each with its own asymptotics and characteristic locking phenomena)
4. **Example:** Classical MITC4-S element attempts to cover these subproblems simultaneously.
5. **Can DPG better adapt to these subproblems simultaneously?**

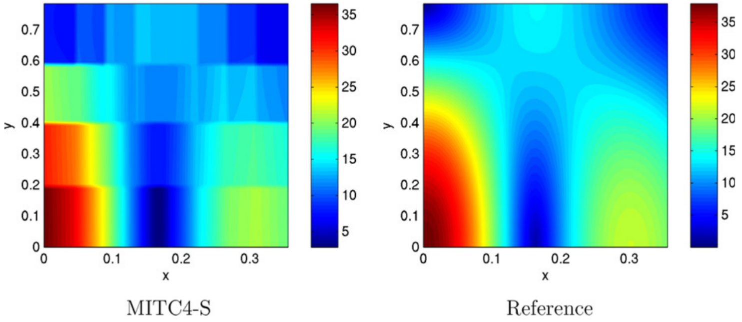

Figure: Comparison of MITC4-S to reference solution.

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