DPG methods for time-harmonic wave propagation in 1D

Ludwig Mitter

Johannes Kepler University Linz

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Overview

Section 1

Introduction

Reminder: Roadmap to DPG

1. Given well-posed BVP (Babuška-Aziz)

$$\text{Find } u \in U : b(u,v) = l(v) \text{ for } \forall v \in V$$

2. Choose trial subspace $U_h = \operatorname{span}\{e_j\} \subset U$ with good approx. props. (Céa)
3. Approximately compute optimal test space: Find $T_h : U_h \mapsto \tilde{V}_h$, where $\tilde{V}_h \subset V$ DG (not global!) "computationally convenient" such that

$$(T_h u_h, \tilde{v}_h)_V = b(u_h, \tilde{v}_h) \text{ for } \forall \tilde{v}_h \in \tilde{V}_h$$
$$T_h \text{ is injective on } U_h$$

4. Set $V_h = \operatorname{span}\{t_j\}, t_j := Te_j$ (t_j basis since T_h injective!)
5. Solve symmetric positive definite system (also for asym. b(.,.)!)

Num. approx. issues for wave propagation

- lg. numerics on wave propagation (high frequ.) "polluted", ie. for exact/approx. sols. $u \in U, u_h \in U_h$ one has

$$\frac{\|u-u_h\|_U}{\|u\|_U} \leq C(k) \inf_{w_h \in U_h} \frac{\|u-w_h\|_U}{\|u\|_U}, C(k) = C_1 + C_2 \underbrace{k^\beta}_{!!!} \underbrace{(Kh)^\gamma}_{\text{OK}}$$

- **Typically:** best. approx. error small if kh small (enough elements per wavelength)
- **Typically:** $\beta = 1$
- **Typically:** num. approx. extremely expensive (high frequ. problems)

Motivation

Standard technology results

1. **free of pollution**, ie. $\beta = 0$ for 1D
2. **reduced pollution** ($\beta > 0$) for higher dims.
3. No general knowledge about γ

Why DPG methodology?

Application on 1D wave propagation gives PG-method that is

1. **free of pollution**, ie. $\beta = 0$
2. AND has $\gamma = 0$

Section 2

Petrov-Galerkin method with optimal test norm

Abstract setting

For **this presentation** assume the **real** setting

- U, V Hilbert spaces
- $(u, v) \in U \times V \mapsto b(u, v) \in \mathbb{R}$ cont. bilinear form
- Cont. linear form $l \in V^*$
- **Abstract variational problem**

$$\text{Find } u \in U : b(u, v) = l(v) \text{ for } \forall v \in V \tag{1}$$

- Operator notation

$$B : U \to V^* \text{ such that } Bu(v) = b(u, v) \text{ for } \forall u \in U, v \in V$$
$$B^* : V \to U^* \text{ such that } B^*v(u) = b(u, v) \text{ for } \forall u \in U, v \in V$$

- **Assume** B **bijection** with cont. inverse

$$B^{-1} : V^* \to U \quad (\text{Reminder: } (B^*)^{-1} = (B^{-1})^*)$$

The optimal test space norm

Definition (Optimal test norm)

$$\|v\|_V := \sup_{u \in U} \frac{|b(u,v)|}{\|u\|_U} \text{ for } \forall v \in V$$

- B^* bijection $\implies$ $\|.\|_V$ equivalent norm on V
- $\|.\|_V$ generated by inner product

$$(w,v)_V := b(R_U^{-1} B^* w, v)$$

with $R_U : U \to U^*$ **Riesz operator**

The optimal test functions

- Let $U_N = \text{span}\{e_j : j = 1, \cdots, N\} \subset U$ fin.-dim.
- Define $T : U \to V$ via

$$(Tu, v)_V = b(u, v) \text{ for } \forall v \in V$$

- **Trial basis** function e_j
- **Optimal test basis** function $t_j := Te_j \in V$
- **Optimal discrete test space**
  $V_N := \text{span}\{t_j : j = 1, \cdots, N\} \subset V$

The optimal test functions

- **PG-scheme** for (1)

$$\text{Find } u_N \in U_N : b(u_N, v_N) = l(v_N) \text{ for } \forall v_N \in V_N \tag{2}$$

- From previous presentations:

Lemma ([DemGop, 2011])

$$\|u - u_N\|_E = \inf_{w_N \in U_N} \|u - w_N\|_E$$

$$\|u\|_E := \sup_{v \in V} \frac{|b(u, v)|}{\|v\|_V}$$

The optimal test functions

- Since **optimal test norm** $\|.\|_V$ in def. of $\|.\|_E$ one has

Theorem

$$\|u\|_E = \|u\|_U \;\; \textit{for}\; \forall u \in U$$
$$\left(\implies \quad \|u - u_N\|_U = \inf_{w_N \in U_N} \|u - w_N\|_U \;\; \textit{for}\; \forall u \in U \right)$$

The error representation function

- (2) is sym. pos. def.
- **Error** $e_N := u - u_N$ can be computed for given u_N

$$\text{Find } Te_N \in V : (Te_N, v)_V = b(u - u_N, v) = l(v) - b(u_N, v) \text{ for } \forall v \in V$$

- **Therefore:** $\|e_N\|_U \overset{\text{thm.}}{=} \|e_N\|_E = \|Te_N\|_V$
- Authors call Te_N error representation function

Equivalent test norms

- $\|.\|_V$ **inconvenient** for **practical computations**
- **IDEA:** Replace $\|.\|_V$ with equivalent norm $\|.\|_{\tilde{V}}$
- Assume $\|.\|_{\tilde{V}}$ generated by computable inner prod. $(.,.)_{\tilde{V}}$
- **New PG-scheme** with **solution** $\tilde{u}_N$
- Now: $\tilde{u}_N$ best approximation wrt.

$$\|u\|_{\tilde{E}} := \sup_{v \in V} \frac{|b(u,v)|}{\|v\|_{\tilde{V}}}$$

- ig. $\|u\|_{\tilde{E}} \neq \|u\|_U$ **BUT**

Equivalent test norms

Theorem

Assume $C_1, C_2 > 0$ *with*

$$C_1 \|v\|_{\tilde{V}} \leq \|v\|_V \leq C_2 \|v\|_{\tilde{V}} \ \text{for } \forall v \in V$$

Then one has

$$\|u - \tilde{u}_N\|_U \leq \frac{C_2}{C_1} \inf_{w_N \in U_N} \|u - w_N\|_U$$

WANT: C_1/C_2 to be (by designing $\|.\|_{\tilde{V}}$)

1. as small as possible
2. independent of problem parameters (eg. wavenumber k)

Practicalities

Roadmap

1. Application of T globally does not yield practical method
2. $\implies$ **DPG-method**: Application of T **locally**
   - V are discont. functs.
   - $(.,.)_V$ locally computable
3. $\implies$ **Approximate local problems** "suitably" by T_N

Practicalities

DPG methodology

- Assume partitioning of **computational domain** Ω into **mesh elements** $\{K\}$
- Test functions in **broken test space**

$$V = V_{DPG} = \prod_K V(K)$$

- **Optimal** $(.,.)_V$ ig. NOT local
- **FIND:** "localizable" $(.,.)_{\tilde{V}}, \|.\|_{\tilde{V}}$ such that

$$\|v\|^2_{\tilde{V}} = \sum_K \|v_K\|^2_{\tilde{V}} \text{ for } \forall v \in V$$

- ($\Longrightarrow$ LOCALIZED error representation fnct. for *adaptivity*)

Section 3

A model time-harmonic transport problem

Model time-harmonic transport problem

Simplified 1D time-harmonic wave propagation problem

$$\begin{aligned} ikp + p' &= 0 \text{ in } (0,1) \\ p(0) &= p_0 \end{aligned}$$

Exact solution

$$p(x) = p_0 e^{-ikx}$$

STEP 1: Conforming V-setting (*spectral mehtod*)

- $V := \mathscr{H}^1(0,1)$
- $U := \mathscr{L}^2(0,1) \times \mathbb{C}$
- **Variational formulation**

$$\text{Find } (p,\hat{p}) \in U : b\Big((p,\hat{p}),q\Big) = p_0\overline{q(0)} \text{ for } \forall q \in V \qquad (3)$$

where

$$b\Big((p,\hat{p}),q\Big) := -\int_0^1 p(\overline{ikq+q'}) + \hat{p}\overline{q(1)}$$

- Flux unknown $\hat{p} \in \mathbb{C}$

STEP 1: Optimal norm and inner product

- **Choose:**

$$\|(p,\hat{p})\|_U^2 := \|p\|_{0,(0,1)}^2 + |\hat{p}|^2$$

- **Optimal test norm:**

$$\|q\|_V = \sup_{(p,\hat{p})\in U} \frac{\left|b\Big((p,\hat{p}),q\Big)\right|}{\|(p,\hat{p})\|_U}$$

- One has

$$\|q\|_V^2 = \left\|ikq + q'\right\|_{0,(0,1)}^2 + |q(1)|^2$$

- Inner prod. generating this norm:

$$(q,r)_V = (ikq + q', ikr + r')_{0,(0,1)} + q(1)\overline{r(1)}$$

STEP 1: discretization

- Trial space discretization: $U_N \equiv U_p := \mathbb{P}_p(0,1) \times \mathbb{C}$
- **Test space discretization**: $V_N \equiv V_p$ via T:
  for $\forall e \in U_N$ the function $q = Te \in V$ solves

$$\text{Find } q \equiv Te \in V :$$

$$(ikq + q', ikr + r')_{0,(0,1)} + q(1)\overline{r(1)} = b(e,r) \text{ for } \forall r \in V$$

- **PG-scheme** for (3)

$$\text{Find } u_N \in U_N : b(u_N, v_N) = l(v_N) \text{ for } \forall v_N \in V_N \qquad (4)$$

STEP 1: discretization

- Know **by theorem**:

$$\|u - u_N\|_U = \inf_{w_N \in U_N} \|u - w_N\|_U \text{ for } \forall u \in U$$

- $\implies$ **explicit** computation

$$\begin{aligned} &\|p - p_N\|^2_{0,(0,1)} + |\hat{p} - \hat{p}_N|^2 \\ &= \inf_{(w_N, \hat{w}_N) \in U_N} \|p - w_N\|^2_{0,(0,1)} + |\hat{p} - \hat{w}_N|^2 \\ &= \inf_{w_N \in U_N} \|p - w_N\|^2_{0,(0,1)} \end{aligned}$$

- $\implies$ p_N coincides with $\mathscr{L}^2(0,1)$-orthog. proj. of p in pols.

STEP 2: An intermediate method

- **Now:** U_N discont. functs.
- $\Omega = (0, 1)$
- $0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < \cdots < x_n = 1$
- Set **elements** $K_j := (x_{j-1}, x_j)$
- Prescribe poly. degr. on K_j:

$$L^2_{hp} := \{ w : w\big|_{K_j} \in \mathbb{P}_{p_j}(K_j) \}$$
$$U_N \equiv \check{U}_{hp} := L^2_{hp} \times \mathbb{C}$$

STEP 2: An intermediate method

- Change inner product on V to

$$\text{STEP 1: } (q,r)_V = (ikq + q', ikr + r')_{0,(0,1)} + q(1)\overline{r(1)}$$

$$\text{STEP 2: } (q,r)_{\tilde{V}} = (ikq + q', ikr + r')_{0,(0,1)} + \frac{1}{2}(q,r)_{0,(0,1)}$$

Lemma

$\|.\|_{\tilde{V}}$ *is* ***equivalent*** *to* $\|.\|_V$ *with*
$C_1 = (2-\sqrt{2})^{1/2}$, $C_2 = (2+\sqrt{2})^{1/2}$)

STEP 2: An intermediate method

Definition

Call q **global optimal test function**:$\Longleftrightarrow$ **optimal test function** for $(p, \hat{p}) \in \check{U}_{hp}$, ie. $q \in \mathscr{H}^1(0,1)$ such that

$$(q, r)_{\tilde{V}} = -\int_0^1 p(\overline{ikr + r'}) + \hat{p}\overline{r(1)} \text{ for } \forall r \in \mathscr{H}^1(0,1)$$

- Set $V_N \equiv \check{V}_{hp}$ to span of all glob. opt. test functs for all $(p, \hat{p}) \in \check{U}_{hp}$.
- **Intermediate method** Find $(\check{p}_{hp}, \hat{p}_{hp}) \in \check{U}_{hp}$:

$$-\int_0^1 \check{p}_{hp}(\overline{ikq + q'}) + \hat{p}_{hp}\overline{q(1)} = p_0\overline{q_1(0)} \text{ for } \forall q \in \check{V}_{hp}$$

STEP 2: An intermediate method

Theorem (Error estimate)

$$\|p - \check{p}\|_{0,(0,1)} \leq \underbrace{\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)^{1/2}}_{\text{indep. of } k!} \inf_{w_{hp} \in L^2_{hp}} \|p - w_{hp}\|_{0,(0,1)}$$

STEP 3: The DPG method

- Method of STEP 2 not practical: Optimal test space V_N global problem
- **STEP 3:** $V = V_{DPG} = \prod_{j=1}^{n} V(K_j), V(K_j) := \mathscr{H}^1(K_j)$
- **DPG variational formulation**

Find $(p, \hat{p}) \in U := \mathscr{L}^2(0,1) \times \mathbb{C}^n$ such that

$$\underbrace{-\sum_{j=1}^{n} p(\overline{ikq_j + q_j'}) + \hat{p}_j \overline{[q]_j}}_{=:b\left((p,\hat{p}),q\right)} = p_0 \overline{q_1(0)} \text{ for } \forall q \in V_{DPG}$$

STEP 3: The DPG method

- Jumps:

$$[q]_j = \begin{cases} q_j(x_j) - q_{j+1}(x_j) \text{ if } j = 1, \cdots, n-1 \\ q_n(1) \text{ if } j = n \end{cases}$$

- Fluxes at element interfaces: $\hat{p} = (\hat{p}_1, \cdots, \hat{p}_n)$
- **Choose:**

$$\|(p, \hat{p})\|_U^2 := \|p\|_{0,(0,1)}^2 + \sum_{j=1}^{n} |\hat{p}_j|^2$$

STEP 3: The DPG method

- **Obtain:**

$$\|q\|^2_{V_{DPG}} = \sum_{j=1}^{n} \|ikq_j + q_j'\|^2_{0,K_j} + |[q]_j|^2$$

$$(q, r)_{V_{DPG}} = \sum_{j=1}^{n} (ikq_j + q_j', ikr_j + r_j')_{0,K_j} + [q]_j \overline{[r]_j}$$

- Does **not satisfy the localization property**!
- $\Longrightarrow$ **replace** with norm that does

$$\|q\|^2_{\tilde{V}} = \sum_{j=1}^{n} \|ikq_j + q_j'\|^2_{0,K_j} + \frac{1}{2}\|q_j\|^2_{0,K_j}$$

$$(q, r)_{\tilde{V}} = \sum_{j=1}^{n} (ikq_j + q_j', ikr_j + r_j')_{0,K_j} + \frac{1}{2}(q_j, r_j)_{0,K_j}$$

STEP 3: The DPG method

- **Observe:** Same norm as in STEP 2, when applied to $q \in \mathscr{H}^1(0,1)$
- **Discrete trial space**: $U_{hp} = L^2_{hp} \times \mathbb{C}^n \subset U$
- Optimal test space computed with $(.,.)_{\tilde{V}}$

STEP 3: The DPG method

- Let $L^2_{hp} = \text{span}\{p_l\}$, p_l with support only on ONE element.
- **Basis** of $U_{hp} = \text{span}\{(p_l, \hat{e}_m)\}$, $e_m = (\delta_{im})_{i=1}^n \in \mathbb{C}^n$
- Test functions can now be **LOCALLY computed** $\rightarrow$ local optimal test functions
- p_l supported on $K_j \implies \circledcirc$ **local optimal test function** q for trial basis $(p_l, 0)$ supported on K_j only and satisfies

$$(ikq_j + q_j', ikr_j + r_j')_{0,K_j} + \frac{1}{2}(q_j, r_j)_{0,K_j}$$
$$= -\int_{x_{j-1}}^{x_j} p_l(\overline{ikr + r'}) \text{ for } \forall r \in V(K_j)$$

STEP 3: The DPG method

- $\circledast$ **Local optimal test function** q corresponding to $(0, \hat{e}_j)$ is supported on $K_j \cup K_{j+1}$ only and satisfies for $\forall r \in V_{DPG}$

$$(ikq_j + q_j', ikr_j + r_j')_{0,K_j} + \frac{1}{2}(q_j, r_j)_{0,K_j} = r_j(x_j)$$

$$(ikq_{j+1} + q_{j+1}', ikr_{j+1} + r_{j+1}')_{0,K_{j+1}} + \frac{1}{2}(q_{j+1}, r_{j+1})_{0,K_{j+1}} = -r_{j+1}(x_{j+1})$$

- **SET:** $V_N = V_{hp} = \text{span}\{\text{these optimal test functs.} \quad \circledast \circledcirc\}$
- $V_{hp} \subset V_{DPG}$

STEP 3: The DPG method

The DPG method

Find $(p_{hp}, \hat{p}^{hp}) \in U_{hp}$ such that

$$-\sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} p_{hp}(\overline{ikq_j + q'j}) + \hat{p}_j^{hp}\overline{[q]_j} = p_0\overline{q_1(0)} \text{ for } \forall q \in V_{hp}$$

STEP 3: The DPG method

Lemma

$\check{V}_{hp} \subset V_{hp}$ $(\implies \check{p}_{hp} = p_{hp})$

which immediately implies

Theorem (Error estimate)

$$\|p - p_{hp}\|_{0,(0,1)} \leq \underbrace{\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)^{1/2}}_{\textit{indep. of } k!} \inf_{w_{hp} \in L^2_{hp}} \|p - w_{hp}\|_{0,(0,1)}$$

NUMERICS

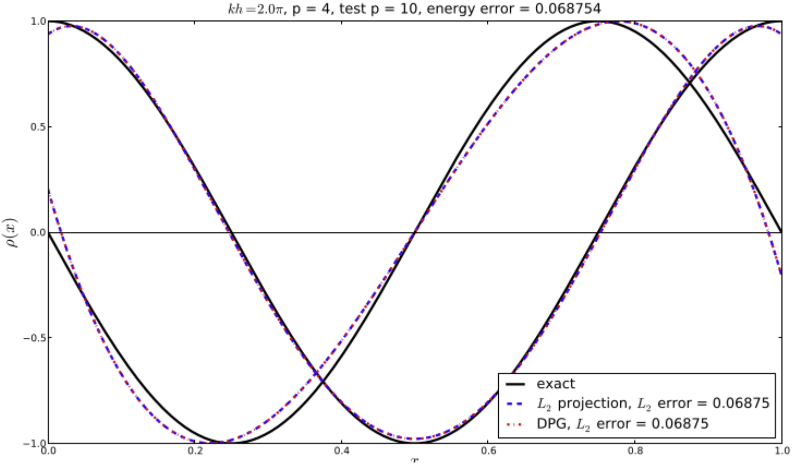
NUMERICS: What the authors did

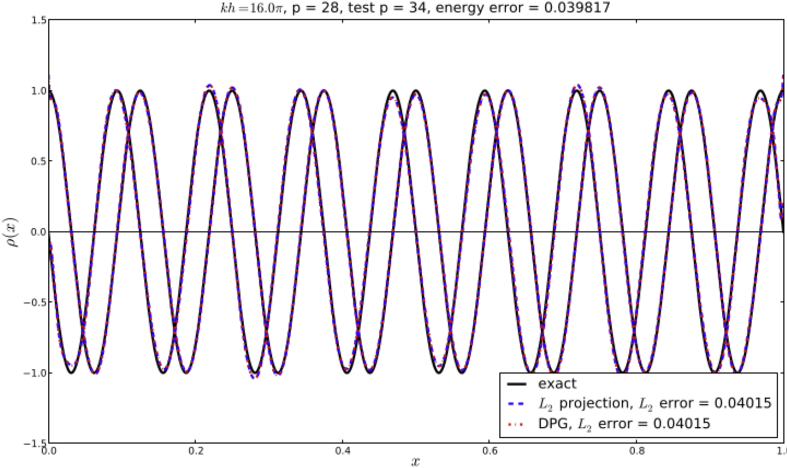
- Higher order approximation approximating **optimal test functions** spanning the discrete test space V_{hp}
- **IC:** $p_0 = 1$

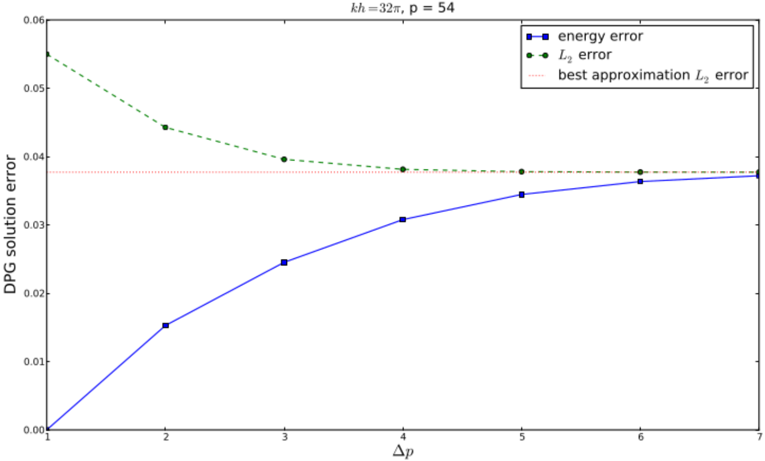
NUMERICS

Example 1: The One-Element-Case

- As expected: no distinction between conforming/DPG methods
- **Observation:** "Higher **enrichment**" gives **better approx. of optimal test func.** (more comp. effort)
  - Cholesky fact. for loc. sys. (effort about $(p + \Delta p)^3$)
  - Comp. cost of loc. sys. negligible compared to glob. sys.


$kh = 2.0\pi$, p = 4, test p = 10, energy error = 0.068754
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0
$\rho(x)$
exact
L_2 projection, L_2 error = 0.06875
DPG, L_2 error = 0.06875

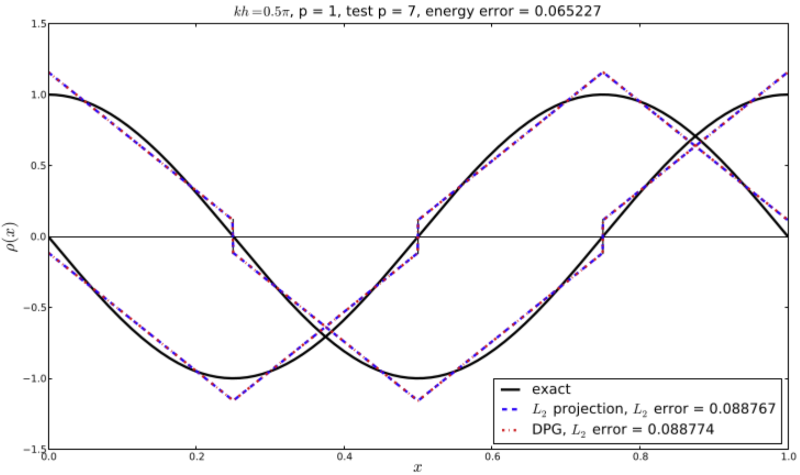

$kh = 16.0\pi$, p = 28, test p = 34, energy error = 0.039817
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
$\rho(x)$
0.0
0.2
0.4
0.6
0.8
1.0
x
exact
L_2 projection, L_2 error = 0.04015
DPG, L_2 error = 0.04015

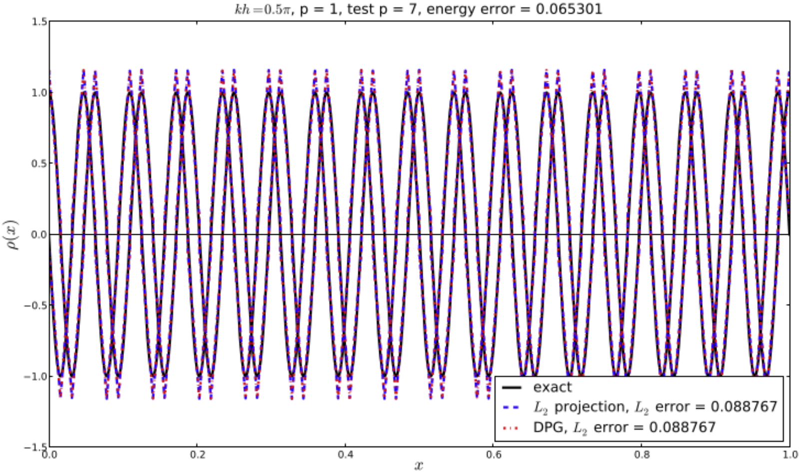

$kh = 32\pi$, p = 54
energy error
L_2 error
best approximation L_2 error
DPG solution error
0.06
0.05
0.04
0.03
0.02
0.01
0.00
1
2
3
4
5
6
7
Δp

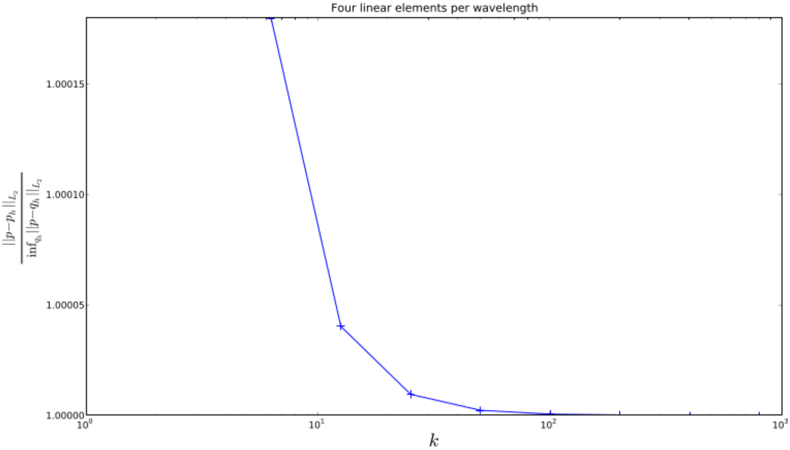
NUMERICS

Example 2: The 4-Element-per-Wavenumber-Case

- **Observation:** very good $\mathscr{L}^2(0,1)$ stability (as indicated by thm., regardless of k)
- Plot: Ratio of DPG error to best approx. error as k is increased
- **Ratio approaches** a k-independent val. (close to 1)


$kh = 0.5\pi$, p = 1, test p = 7, energy error = 0.065227
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.2
0.4
0.6
0.8
1.0
$\rho(x)$
x
exact
L_2 projection, L_2 error = 0.088767
DPG, L_2 error = 0.088774


$kh = 0.5\pi$, p = 1, test p = 7, energy error = 0.065301
$\rho(x)$
x
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.2
0.4
0.6
0.8
1.0
exact
L_2 projection, L_2 error = 0.088767
DPG, L_2 error = 0.088767


Four linear elements per wavelength
$\frac{\|p-p_h\|_{L_2}}{\inf_{q_h}\|p-q_h\|_{L_2}}$
1.00015
1.00010
1.00005
1.00000
10^0
10^1
10^2
10^3
k

Section 4

The Helmholtz model problem

Helmholtz model problem

Coupled 1D Helmholtz problem

$$
\begin{aligned}
ik\frac{p}{c\rho} + u' &= 0 \text{ in } \Omega \\
ikc\rho u + p' &= 0 \in \Omega \\
u(0) &= u_0 \\
p(1) &= c\rho u(1)
\end{aligned}
$$

Exact solution

$$
u(x) = u_0 e^{-ikx} \quad p(x) = c\rho u(x)
$$

STEP 3: The DPG method

Theorem

$$\|u - u_{hp}\|^2_{0,(0,1)} + \|p - p_{hp}\|^2_{0,(0,1)} \leq C \inf_{w_{hp}, s_{hp} \in L^2_{hp}} \|u - w_{hp}\|^2_{0,(0,1)} + \|p - s_{hp}\|^2_{0,(0,1)}$$

$$C = \left(\sqrt{\frac{5+\sqrt{5}}{2}} + \sqrt{\frac{3+\sqrt{5}}{2}} \right)^2$$ *indep. of k!*

Section 5

Conclusion

This presentation answers:

1. How does one design a norm on the test space V to minimize the discretization error in a given trial norm on U? (also for multi.dim.)
2. How to localize the resultant test norm without losing uniform stability? (also for multi.dim.)

DPG Method COMPETITIVE?

- Burden of dealing with small parameter has been moved to the problem of finding the optimal test functions
- **DPG extremely stable** compared to trad. techniques
- DPG introduces add. dofs fluxes: n elements of order p per wavelenth for domain of m wavelengths
  - DPG: $2(p+1)mn + 2mn$ dofs (stat. cond. $2mn$ dofs)
  - Conforming: pmn (stat. cond. mn dofs)
- $\Longrightarrow$ DPG competitive for **large wavenumbers only**!

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