Analysis of the practical DPG Method

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Overview

Ideal Testfunctions

For the problem: find $u \in U$ satisfying:

$$b(u, v) = l(v) \ \ \forall v \in V$$

the opterator $T : U \mapsto V$ is defined as:

$$(Tw, v)_V = b(w, v) \ \ \forall v \in V$$

for a chosen finite dimensional subspace U_h of the trial space U the corresponding ideal test functions are defined as follows:

$$V_h = T(U_h)$$

Ideal/Practical DPG Method

This leads to the ideal DPG method:

$$\text{find} \quad u_h \in U_h : \quad b(u_h, v) = f(v) \;\; \forall v \in V_h = T(U_h)$$

computing V_h leads to an infinite dimensional problem. A numerically cheaper approach is to first approximate V by a finite dimensional subspace V^r. We approximate T by T^r:

$$(T^r w, v)_V = b(w, v) \;\; \forall v \in V^r$$

now the corresponding nearly optimal test functions are

$$V_h^r = T^r(U_h)$$

And this leads to the practical DPG method:

$$\text{find} \quad u_h^r \in U_h : \quad b(u_h^r, v) = l(v) \;\; \forall v \in V_h^r = T^r(U_h)$$

Analysis in the abstract setting

Theorem Under the assumptions:

$$\{w \in U : b(w,v) = 0 \;\; \forall v \in V\} = \{0\} \tag{1}$$

$$\exists C_1 : C_1 ||v||_V \leq sup_{w \in U} \frac{b\,(w,v)}{||w||_U}, \;\; \forall v \in V \tag{2}$$

$$\exists C_2 : b(w,v) \leq C_2 ||w||_U ||v||_V \tag{3}$$

and the existence of a linear operator $\Pi : V \mapsto V^r$ such that:

$$b(w, v - \Pi v) = 0 \;\; \forall w \in U_h \tag{4}$$

$$||\Pi v||_V \leq C_\Pi ||v||_V \tag{5}$$

the problem is well posed and

$$||u - u_h^r||_U \leq \frac{C_2 C_\Pi}{C_1} inf_{w \in U_h} ||u - w||_U$$

Proof:logical structure

$$C_1||v||_V \leq sup_{w\in U}\frac{b\left(w,v\right)}{||w||_U} \implies C_1||w||_U \leq sup_{v\in V}\frac{b\left(w,v\right)}{||v||_V}$$

$$C_1||w||_U \leq sup_{v\in V}\frac{b\left(w,v\right)}{||v||_V} \text{ and } \Pi \implies \frac{C_1}{C_\Pi}||w||_U \leq sup_{v\in V^r}\frac{b\left(w,v\right)}{||v||_V}$$

$$\frac{C_1}{C_\Pi}||w||_U \leq sup_{v\in V^r}\frac{b\left(w,v\right)}{||v||_V} \text{ and } T^r \implies \frac{C_1}{C_\Pi}||w||_U \leq sup_{v\in V_h^r}\frac{b\left(w,v\right)}{||v||_V}$$

With (1), (3) and the last line the conditions of Babuska Aziz are fullfilled and the result follows.

Some remarks

- Under the assumptions of the previous theorem the operator $T^r : U_h \mapsto V^r$ is injective. That means:

$$dim(V_h^r) = dim(U_h).$$

- One can also think about the condition number of the stiffness matrix. Let $\mathcal{B}_i$ be a basis for U_h. Moereover let $\bar{x} = \sum_i x_i \mathcal{B}_i$ the basis expansion of any $\bar{x}$. If now λ_0, λ_1 are positive number such that:

$$\lambda_0 ||x||_{l^2}^2 \leq ||\bar{x}||_U \leq \lambda_1 ||x||_{l^2}^2 \quad \forall \bar{x} \in U_h$$

The spectral condition number $\kappa(S)$ can be estimated:

$$\kappa(s) \leq \frac{\lambda_1}{\lambda_0} \frac{C_2^2 C_\Pi^2}{C_1^2}$$

Infinite dimensional spaces

For $\mathbb{V} = \mathbb{R}^n$, we denote $L^2(\Omega, \mathbb{V})$ the vector valued function whose components are in $L^2(\Omega)$. We set the trial and test spaces as:

$$U = L^2(\Omega, \mathbb{V}) \times L^2(\Omega) \times H_0^{\frac{1}{2}}(\partial\Omega_h) \times H^{-\frac{1}{2}}(\partial\Omega_h)$$

$$V = H(div, \Omega_h) \times H^1(\Omega_h)$$

where

$$H_0^{\frac{1}{2}}(\partial\Omega) := \{\eta \in \Pi_K H^{\frac{1}{2}}(\partial K) : \exists w \in H_0^1(\Omega) : \quad \eta|_{\partial K} = w|_{\partial K} \quad \forall K \in \Omega_h\}$$

$$H^{-\frac{1}{2}}(\partial\Omega) := \{\eta \in \Pi_K H^{-\frac{1}{2}}(\partial K) : \exists q \in H(div, \Omega) : \ \eta|_{\partial K} = q \cdot n|_{\partial K} \ \forall K \in \Omega_h\}$$

Norms

$$||(\sigma, u, \hat{u}, \hat{\sigma}_n)||_U^2 = ||\sigma||_\Omega^2 + ||u||_\Omega^2 + ||\hat{u}||^2_{H^{\frac{1}{2}}(\partial\Omega)} + ||\hat{\sigma}||^2_{H^{-\frac{1}{2}}(\partial\Omega)}$$

Here the traces must come from a global $H^1(\Omega)$ and the fluxes from a global $H(div, \Omega)$ function.

$$||(\tau, v)||_V^2 = ||\tau||^2_{H(div,\Omega_h)} + ||v||^2_{H^1(\Omega_h)}$$

Ultraweak formulation

Find $\bar{u} := (\sigma, u, \hat{u}, \hat{\sigma}_n) \in U$ such that:

$$b(\bar{u}, \bar{v}) = l(\bar{v}) \ \ \forall \bar{v} = (\tau, v) \in V$$

,with

$$b(\bar{u}, \bar{v}) = (\sigma, \tau)_\Omega - (u, div\ \tau)_{\Omega_h} + \langle \hat{u}, \tau \cdot n \rangle_{\partial\Omega_h} - (\sigma, grad\ \ v)_{\Omega_h} + \langle v, \hat{\sigma_n} \rangle_{\partial\Omega_h}$$

$$l(\bar{v}) = (f, v)_\Omega$$

Finite dimensional spaces

We first define:

$$P_p(\partial K) := \{\mu : \mu|_F \in P_p(F) \ \ \forall F \in \Delta_{n-1}(K)\}$$

where $\Delta_{n-1}(K)$ are the $n-1$ dimensional simplexes of K

$$\tilde{P}_p(\partial K) := P_p(\partial K) \cap C^0(\partial K)$$

Finite dimensional spaces

Now we can define $U_h := \{(\sigma, u, \hat{u}, \hat{\sigma}_n) \in U :$

$$\begin{aligned} \sigma|_K &\in P_p(K, \mathbb{V}) \\ u|_K &\in P_p(K) \\ \hat{u}|_{\partial K} &\in \tilde{P}_{p+1}(\partial K) \\ \hat{\sigma}_n|_{\partial K} &\in P_p(\partial K) \end{aligned}$$

$$\forall \quad K \in \Omega_h\}$$

Finite dimensional spaces

The only thing left to be choosen is

$$V^r := \{(\tau, v) \in V : \tau|_K \in P_r(K, \mathbb{V}), v|_K \in P_r(K) \quad \forall K \in \Omega_h\}$$

with $r \geq p + N$

Our job now is to check all the conditions of our theorem. The first three conditions were already shown, because they are sufficient for the ideal DPG method. Therefore, we only have to construct the operator Π. This operator will be constructed in the form:

$$\Pi \bar{v} = (\Pi^{div}_{p+2}\tau, \Pi^{grad}_{r} v)$$

Analysis of the Laplace equation

Theorem

Let $r \geq p + N$ *then for every* $v \in H^1(K)$ *there is a unique* $\Pi_r^0 v \in B_r^{grad}(K) := \{p_r \in P_r(K) : p_r|_E = 0 \;\; \forall E \in \Delta_{n-2}(K)\}$ *satisfying:*

$$(\Pi_r^0 v - v, q_{p-1})_K = 0 \;\; \forall q_{p-1} \in P_{p-1}(K)$$
$$(\Pi_r^0 v - v, \mu_p)_{\partial K} = 0 \;\; \forall \mu_p \in P_p(\partial K)$$
$$||\Pi_r^0 v||_K + h_K ||grad \; \Pi_r^0 v||_K \leq C(||v||_K + h_K ||grad \;\; v||_K)$$

Analysis of the Laplace equation

Theorem

Let $r \geq p + N$. *Define* $\Pi_r^{grad} v = \Pi_r^0(v - \bar{v}) + \bar{v}$, *where* $\bar{v}|_K = |K|^{-1} \int_K v$.
Then:

$$(\Pi_r^{grad} v - v, q_{p-1})_K = 0 \quad \forall q_{p-1} \in P_{p-1}(K)$$

$$(\Pi_r^{grad} v - v, \mu_p)_{\partial K} = 0 \quad \forall \mu_p \in P_p(\partial K)$$

$$||\Pi_r^{grad} v||_{H^1(K)} \leq C||v||_{H^1(K)}$$

Analysis of the Laplace equation

Theorem

There exists an operator $\Pi^{div}_{p+2} : H(div, K) \mapsto P_{p+2}(K, \mathbb{V})$ *such that for every* $\tau \in H(div, K)$*, we have:*

$$(\Pi^{div}_{p+2}\tau, q_p)_K = (\tau, q_p)_K \quad \forall q_p \in P_p(K\mathbb{V})$$

$$\langle \Pi^{div}_{p+2}\tau \cdot n, \mu_{p+1}\rangle_{\partial K} = \langle \mu_{p+1}, \tau \cdot n\rangle_{\frac{1}{2},\partial K} \quad \forall \mu_{p+1} \in \tilde{P}_{p+1}(\partial K)$$

$$||\Pi^{div}_{p+2}\tau||_{H(div,K)} \leq C||\tau||_{H(div,K)}$$

This proves the conditions of our general theorem and therefore the approximation is as good as the bestapproximation.

The discrete space V^r can also be chosen a little bit weaker namely the previous results can also be obtained by using:

$$V_r = \{(\tau, v) \in V : \tau_K \in P_{p+2}(K, \mathbb{V}), v|_K \in P_{p+N}(K), \quad \forall \in \Omega_h\}$$

For h being the maximal diameter of all K one can show

$$||\bar{u} - \bar{u}_h||_U \leq Ch^s \left(||u||_{H^{s+1}(\Omega)} + ||\sigma||_{H^{s+1}(\Omega)}\right)$$

for all $\frac{1}{2} < s \leq p + 1$.

Remarks and Conclusions

Proof: By the abstract theorem we obtain:

$$||\bar{u} - \bar{u}_h||_U \leq \frac{C_2 C_\Pi}{C_1} inf_{w \in U_h} ||\bar{u} - w||_U$$

with

$$||\bar{u} - w||_U = \underbrace{||\sigma_{\bar{u}} - \sigma_w||^2_\Omega}_{\text{standard}} + \underbrace{||u_{\bar{u}} - u_w||^2_\Omega}_{\text{standard}} + \underbrace{||\hat{u}_{\bar{u}} - \hat{u}_w||^2_{H^{\frac{1}{2}}(\partial\Omega)}}_{\text{not standard}} + \underbrace{||\hat{\sigma}_{\bar{u}} - \hat{\sigma}_w||^2_{H^{-\frac{1}{2}}(\partial\Omega)}}_{\text{not standard}}$$

Remarks and Conclusions

Let Ω_h be a quasiuniform tetrahedral mesh, r as above. Then the spectral condition number of the stiffness matrix S of the DPG method can be estimated by:

$$\kappa(S) \leq Ch^{-2}$$

Literature

- J. Gopalakrishnan and W. Qiu. An analysis of the practical DPG method. Math. Comput., 83(286):537-552, 2014.