The DPG Method for the Stokes Problem

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The Stokes Problem

Problem (Strong form)

$$
\begin{aligned}
-\triangle \boldsymbol{u} + \nabla p &= f && \text{in } \Omega \\
\nabla \cdot \boldsymbol{u} &= g && \text{in } \Omega \\
\boldsymbol{u} &= \boldsymbol{u_D} && \text{on } \partial\Omega
\end{aligned}
$$

The Stokes Problem

Problem (First order system)

$$
\begin{aligned}
-\nabla \cdot \sigma + \nabla p &= f && \textit{in } \Omega \\
\nabla \cdot \boldsymbol{u} &= g && \textit{in } \Omega \\
\sigma &= \nabla \boldsymbol{u} && \textit{in } \Omega \\
\boldsymbol{u} &= \boldsymbol{u_D} && \textit{on } \partial\Omega
\end{aligned}
$$

- well-posedness later . . .

Classical Methods

- Special element choices to satisfy discrete inf-sup conditions
- MINI, Taylor-Hood, ...
- Often: equal order elements for $\boldsymbol{u}, p$

  $\rightarrow$ suboptimal convergence for p
- Finding good spaces can be difficult

  $\rightarrow$ DPG choses spaces automatically

Strong Formulation

Using $u = \{\boldsymbol{u}, p, \sigma\}$ and $v = \{\boldsymbol{v}, q, \tau\}$, we can write Stokes as

$$Au = f$$
$$\boldsymbol{u} = \boldsymbol{u_D}$$

with

$$Au = \{-\text{div}(\sigma - pI), \text{div } \boldsymbol{u}, \sigma - \nabla \boldsymbol{u}\}$$

and $f = \{\boldsymbol{f}, g, 0\}$

Weak Formulation

Strong form:

$$Au = \{-\text{div}(\sigma - pI), \text{div}\ \boldsymbol{u}, \sigma - \nabla \boldsymbol{u}\}$$

Integration by parts and summing up yields:

$$(Au, v) = (u, A^* v) + c(u, v)$$

with

$$A^* v = \{\text{div}(\tau - qI), -\text{div}\ \boldsymbol{v}, \tau + \nabla \boldsymbol{v}\},$$

and

$$c(u, v) = \langle -\sigma + pI, v \rangle + \langle u, -\tau + qI \rangle$$

the boundary terms from partial integration.

Graph Energy Spaces

Graph energy spaces:

$$H_A(\Omega) := \{u \in L^2(\Omega) : Au \in L^2(\Omega)\}$$
$$H_{A^*}(\Omega) := \{v \in L^2(\Omega) : A^* v \in L^2(\Omega)\}$$

For Stokes:
Not self-adjoint, but energy graph spaces are identical:

$$H_A(\Omega) = \{\{\boldsymbol{u}, p, \sigma\} \in \boldsymbol{L^2} \times L^2 \times L^2 : \sigma - pI \in H(\mathsf{div}, \Omega), \boldsymbol{u} \in H^1(\Omega)\}$$
$$H_{A^*}(\Omega) = \{\{\boldsymbol{v}, q, \tau\} \in \boldsymbol{L^2} \times L^2 \times L^2 : \tau - qI \in H(\mathsf{div}, \Omega), \boldsymbol{v} \in H^1(\Omega)\}$$

Same trace operators:

$$\mathsf{tr}_A : H_A(\Omega) \ni \{\boldsymbol{u}, p, \sigma\} \mapsto \{(-\widehat{\sigma + pI})n, \hat{\boldsymbol{u}}\} = \{\hat{t}, \hat{\boldsymbol{u}}\} \in H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$$

Some Generalizations

Allow possibly different trace operators / spaces:

$$\mathrm{tr}_A : H_A(\Omega) \to \hat{H}_A(\Gamma), \qquad u \mapsto \hat{u} \text{ surjective}$$
$$\mathrm{tr}_{A^*} : H_{A^*}(\Omega) \to \hat{H}_{A^*}(\Gamma), \qquad v \mapsto \hat{v} \text{ surjective}$$

with min. energy extension norm.

Extend previous formulation to:

$$(Au, v) = (u, A^* v) + c(\hat{u}, \hat{v})$$

with $u \in H_A(\Omega), v \in H_{A^*}(\Omega), \hat{u} \in \hat{H}_A(\Gamma), \hat{v} \in \hat{H}_{A^*}(\Gamma)$

Boundary Terms

Stokes boundary term:

$$c(\hat{u},\hat{v}) = c(\{\hat{t},\hat{\boldsymbol{u}}\},\{\hat{r},\hat{\boldsymbol{v}}\}) = \langle \hat{\boldsymbol{u}},\hat{r}\rangle + \langle \hat{t},\hat{\boldsymbol{v}}\rangle$$

Assume

$$c(\hat{u},\hat{v}) = \langle C\hat{u},\hat{v}\rangle = \langle C_1\hat{u},\hat{v}\rangle + \langle C_2\hat{u},\hat{v}\rangle = \langle C_1\hat{u},\hat{v}\rangle + \langle \hat{u},C_2'\hat{v}\rangle$$

with "reasonable" (closed range) operators C_1, C_2.

For Stokes: $C_1\hat{u} = \{0,\hat{\boldsymbol{u}}\}$, $C_2\hat{u} = \{\hat{t},0\}, C_2'\hat{v} = \{0,\hat{\boldsymbol{v}}\}$

Boundary Terms

Lemma

If

$$\big(\langle \hat{u}, C_2' \hat{v} \rangle = 0 \;\; \forall \hat{u} : C_1 \hat{u} = 0\big) \Rightarrow C_2' \hat{v} = 0$$

then we have the splitting

$$\hat{H}_A(\Gamma) = \mathcal{N}(C_2) \oplus \mathcal{N}(C_1) =: \hat{H}_A^1(\Gamma) \oplus \hat{H}_A^2(\Gamma)$$

$\rightarrow$ unique splitting $\hat{u} = \hat{u}_1 + \hat{u}_2$.

Back to the Strong Formulation

Assumptions:

- compatibility: $(f, v) - \langle f_D, \hat{v} \rangle = 0 \quad \forall v : A^* v = 0, \quad C_2' \hat{v} = 0$
- closed range: $\|Au\| \geq \gamma \|u\| \quad \forall u \in \mathcal{N}(A)^\perp$

Theorem

If these assumptions are fulfilled, the strong problem

$$Au = f$$

$$C_1 \hat{u} = f_D \text{ on } \Gamma$$

has a unique solution $u \in \mathcal{N}(A)^\perp$ *and*

$$\tilde{\gamma} \|u\|_{H_A(\Omega)} \leq \left(\|f\|^2 + \|f_D\|^2 \right)^{\frac{1}{2}}$$

holds (analogous result for (A^*, C_2')*).*

Back to the Strong Formulation

Proof (Roadmap).

- define $b(u,(w_1,\hat{w}_2)) := (Au, w_1) - \langle C_1\hat{u}, w_2\rangle$
  with $u \in H_A(\Omega), w_1 \in L^2(\Omega), \hat{w}_2 \in W_2 := \mathcal{N}(C_2') \cap H_{A^*}(\Gamma)$
- equivalent problem: $b(u,(w_1,\hat{w}_2)) = (f, w_1) - \langle f_D, \hat{w}_2\rangle$
- show that
  $\mathcal{N}(B') = \{(w_1,\hat{w}_2) : A^*w_1 = 0, C_2'w_1 = 0, \mathrm{tr}(w_1) = \hat{w}_2\}$
- show inf-sup

□

Ultra-Weak Formulation

Problem

Find $u \in L^2(\Omega), \hat{u}_2 \in \hat{H}^2_A(\Gamma)$ *such that*

$$(u, A^* v) + \langle \hat{u}_2, C_2' \hat{v} \rangle = (f, v) - \langle f_D, \hat{v} \rangle \quad \forall v \in H_{A^*}(\Omega)$$

- $\hat{u}_2$ now defined as new independent unknown
- no boundary conditions for test-functions v

Theorem (Well-Posedness)

Follow from previous theorem, since $b \sim (A^*, C_2')$

DPG Formulation

- Globally conforming $u \in H_A(\Omega)$
- Broken test-function $v \in H_{A^*}(\Omega_h)$

$$\sum_K (Au, v)_K = \sum_K (u, A^* v)_K + \sum_K c_{\partial K}(\hat{u}, \hat{v})$$
$$\Leftrightarrow: (Au, v) = (u, A_h^* v)_h + c_h(\hat{u}, \hat{v})$$

Homogenization

Split $u = E(\mathrm{tr}(u)) + \tilde{u}$
with

- an extension of the trace $E(\mathrm{tr}(u)) \in H_A$ (e.g. minimum norm extension)
- homogeneous part $\tilde{H}_A := \{u \in H_A(\Omega) : \mathrm{tr}(u) = 0 \text{ on } \Gamma\}$
- similar for traces in $\hat{H}_A(\Gamma_h)$: $\hat{u} = \hat{E}\hat{u}_0 + \hat{\tilde{u}}$
(with $\hat{u}_0 = \hat{u}|_\Gamma$)

With the minimal norm extension, we have

$$\hat{H}_A(\Gamma_h) = \hat{E}\hat{H}_A(\Gamma) \oplus \hat{\tilde{H}}_A(\Gamma_h)$$

Homogenization

Incorporate known boundary data:

$$
\begin{aligned}
c_h(\hat{u},\hat{v}) &= c_h(\hat{E}\hat{u}_0,\hat{v}) + c_h(\hat{\tilde{u}},\hat{v}) \\
&= c_h(\hat{E}\hat{u}_0^1,\hat{v}) + c_h(\hat{E}\hat{u}_0^2,\hat{v}) + c_h(\hat{\tilde{u}},\hat{v}) \\
&= \langle f_D,\hat{v}\rangle + c_h(\hat{E}\hat{u}_0^2,\hat{v}) + c_h(\hat{\tilde{u}},\hat{v})
\end{aligned}
$$

Problem

Find $u \in L^2(\Omega), \hat{u} \in \hat{E}\hat{H}_A^2(\Gamma) \oplus \hat{\tilde{H}}_A(\Gamma_h)$ *such that*

$$
(u, A_h^* v)_h + c_h(\hat{u},\hat{v}) = (f,v) - \langle f_D,\hat{v}\rangle_{\Gamma_h} \qquad \forall v \in H_{A^*}(\Omega_h)
$$

Well-Posedness

Theorem

The previous problem is well-posed and has a solution $(u, \hat{u})$ *coinciding with the ultra-weak solution. Furthermore,*

$$\sup_{v \in H_{A^*}(\Omega_h)} \frac{|(u, A_h^* v)_h + c_h(\hat{u}, \hat{v})|}{\|v\|_{H_{A^*}(\Omega_h)}} \geq \gamma_{DPG} \left(\|u\|^2 + \|\hat{u}\|_{\hat{H}_A(\Gamma_h)} \right)^{\frac{1}{2}}$$

for all $\hat{u} \in \hat{E}\hat{H}_A^2(\Gamma) \oplus \hat{\tilde{H}}_A(\Gamma_h)$ *and* $u \in L^2$ *orthogonal to the null space* $\{(u, \hat{u}) : u \in \mathcal{N}(A|_U), \hat{u} = u$ *on* $\Gamma_h\}$ *with mesh-independent* γ_{DPG}.

Well-Posedness

Assumptions:

- $\mathcal{R}(A^*) = L^2(\Omega)$ (surjective)
- f, f_D satisfy the compatibility condition
  $(f, v) = \langle f_D, \hat{v} \rangle = 0 \quad \forall v \in \mathcal{N}(A^*|_V)$
- existence of energy trace spaces
- $c(u, v)$ is definite
- suitable splitting into $C_1 + C_2$
- lower bound for A in the homogeneous case in $\mathcal{N}(A)^\perp$

$\Rightarrow$ the Stokes problem satisfies all these conditions

Numerical Experiments

- $L^2(\Omega), H^{-\frac{1}{2}}(\Gamma) \leftrightarrow \mathbb{P}_k,\ H^{\frac{1}{2}}(\Gamma) \leftrightarrow \mathbb{P}_{k-1}$
- if u sufficiently smooth

$$\|u - w_h\| \leq C_1 h^{k+1}$$

(for exact optimal test-functions)

- optimal test-functions via enrichment $k_{test} = k + 2$
- test norms:

$$\|v\|_{graph}^2 = \|v\|^2 + \|\nabla \cdot \tau - \nabla q\|^2 + \|\nabla \cdot v\|^2 + \|\tau + \nabla v\|^2$$
$$\|(\tau, \boldsymbol{v}, q)\|_{naive}^2 = \|v\|^2 + \|\nabla \cdot \tau\|^2 + \|\nabla \boldsymbol{v}\|^2 + \|\nabla q\|^2$$

- $V_{naive} \subset V_{graph}$

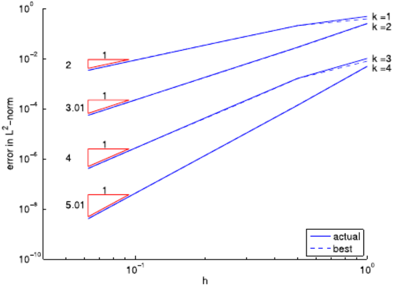
Test 1: Manufactured Solution

Numerical test with predetermined solution:

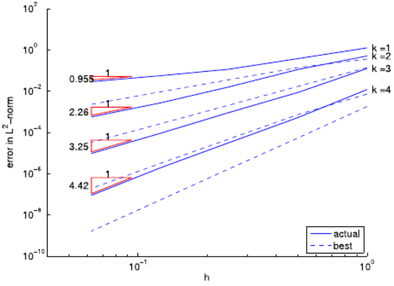
$$\begin{aligned} u_1 &= -e^x(y\cos y + \sin y) \\ u_2 &= e^x y \sin y \\ p &= 2e^x \sin y \end{aligned}$$

on $\Omega = (-1,1)^2$ using uniform quadrilateral meshes.

Convergence Rates - Naive Test Norm



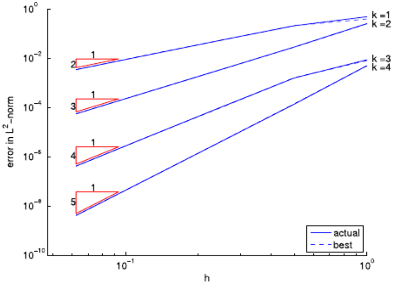
(a) u_1.



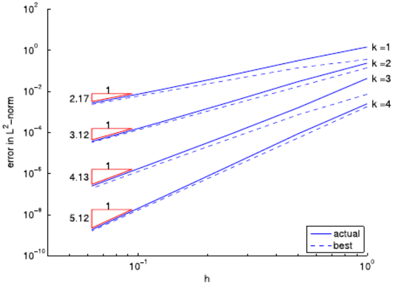
(c) p.

- Optimal rates for $\boldsymbol{u}$ but not for p
- $\boldsymbol{u}$ almost fits L^2 projection
- Optimal test-functions may not be contained in V_{naive}

Convergence Rates - Adjoint Graph Energy Norm



(a) u_1.

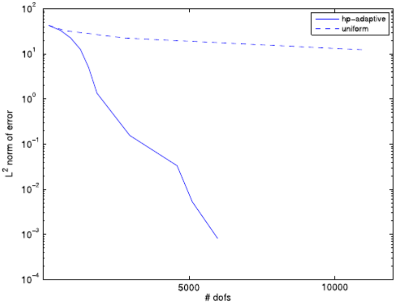


(c) p.

- Optimal rates for $\boldsymbol{u}$ and p
- Both almost fit L^2 projection (proof ?)

hp-Adaptivity: Lid-driven Cavity Flow

- Comparison with overkill mesh (64×64 quintic elements)
- Naive hp-refinement strategy
- Prior knowledge used for refinement



Thank You.

References:

1. N. V. Roberts, T. Bui-Thanh, L. Demkowicz. The DPG method for the Stokes problem. Computers and Mathematics with Applications, volume 67 (2014), 966–995