A Class of Discontinuous Petrov-Galerkin Methods

Optimal Test Functions

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Content

Overview

- Review
- Optimal Test Spaces
- Properties & Relations
- Summary

Abstract Problem

Want to solve

Problem

Find $u \in U$:

$$b(u, v) = l(v) \quad \forall v \in V$$

Operator notation:

Problem

Find $u \in U$:

$$Bu = l$$

with $l \in V^*$, $B : U \to V^*$, $\langle Bu, v \rangle_V = b(u, v)$

Review

Short review of the theorems from

- Lax-Milgram
- Babuška-Aziz
- Céa

Lax-Milgram

Theorem (Lax-Milgram)

If the bilinear form $a : U \times U \to \mathbb{R}$ *satisfies*

(B) $a(u,v) \leq \mu_2 \|u\| \|v\|$

(C) $a(u,u) \geq \mu_1 \|u\|^2$

then the problem

$$a(u,v) = f(v) \qquad \forall v \in U$$

has a unique solution.

Céa's Lemma

Crucial lemma for approximations:

Lemma (Céa)

Assume (B), (C), then

$$\|u - u_h\|_U \leq \frac{\mu_2}{\mu_1} \inf_{w_h \in U_h} \|u - w_h\|_U$$

with the discrete solution u_h, satisfying

$$a(u_h, v_h) = f(v_h) \quad \forall v_h \in U_h \subset U$$

Main message: just a constant factor worse than the best we could do

Abstract Problem

However: want to solve

Problem

Find $u \in U$:

$$b(u, v) = l(v) \quad \forall v \in V$$

with different trial and test spaces.

$\rightarrow$ Lax-Milgram not directly applicable

$\rightarrow$ Generalization: Babuška-Aziz

Theorem of Babuška-Aziz

Theorem (Babuška-Aziz)

The problem $b(u,v) = l(v)$ has a unique solution, if and only if

(B) $b(u,v) \leq \mu_2 \|u\|_U \|v\|_V$

(I) $\inf_{u \in U} \sup_{v \in V} \frac{b(u,v)}{\|u\|_U \|v\|_V} \geq \mu_1$

(U) $\inf_{v \in V} \sup_{u \in U} \frac{b(u,v)}{\|u\|_U \|v\|_V} \geq \mu_1$

Rewrite (U) as $\mathcal{N}(B^*) = \{0\}$

Theorem of Babuška-Aziz

Theorem (Babuška-Aziz)

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(U) $\{v \in V : b(u,v) = 0 \;\; \forall u \in U\} = \{0\}$

Relation to Lax-Milgram: coercivity (C) implies double inf-sup condition (I), (U)

Céa's Lemma in the Babuška-Aziz Setting

We can show a similar result for the assumptions of Babuška-Aziz:

Lemma (Céa)

$\|u - u_h\|_U \leq (1 + \frac{\mu_2}{\mu_1}) \inf_{w_h \in U_h} \|u - w_h\|_U$

Again: just a constant factor worse than the best we could do

Céa's Lemma in the Babuška-Aziz Setting

Lemma

$\|u - u_h\|_U \leq (\textcircled{1} + \frac{\mu_2}{\mu_1}) \inf_{w_h \in U_h} \|u - w_h\|_U$

Again: just a constant factor worse than the best we could do

However: $1 + \frac{\mu_2}{\mu_1}$

Improved Céa's Lemma in the Babuška-Aziz Setting

Lemma (Xu, Zikatanov 2003)

$\|u - u_h\|_U \leq \frac{\mu_2}{\mu_1} \inf_{w_h \in U_h} \|u - w_h\|_U$

Lemma (Kato 1960)

Let $P : H \to H$ with $0 \neq P = P^2 \neq I$. Then it holds:

$$\|P\| = \|I - P\|$$

- Now: same constant as Lax-Milgram
- Further improvements? Not in general.

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Motivation

$$\|u - u_h\|_U \leq \frac{\mu_2^h}{\mu_1^h} \inf_{w_h \in U_h} \|u - w_h\|_U$$

Some observations:

- inf-part: approximation quality of trial space U_h
- $\mu_2^h \geq \mu_1^h$: depends on b, U_h, V_h

Motivation

$$\|u - u_h\|_U \le \frac{\mu_2^h}{\mu_1^h} \inf_{w_h \in U_h} \|u - w_h\|_U$$

Some observations:

- inf-part: approximation quality of trial space U_h
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Idea:

- Fix trial spaces $U_h \subset U$ with good approximation quality
- Chose test space V_h such that $\mu_1^h = \mu_2^h$
- $\Rightarrow$ get the best approximation

Optimal Test Spaces

- For b symmetric, > 0: used the energy norm
- Similar: define energy norm

$$\|u\|_E := \sup_{v \in V} \frac{b(u,v)}{\|v\|_V} = \|Bu\|_{V^*}$$

Lemma (Norm Equivalence)

Iff b satisfies (B),(I) then we have

$$\mu_1 \|u\|_U \leq \|u\|_E \leq \mu_2 \|u\|_U$$

Optimal Test Spaces

- Define map from trial to test space $T : U \to V$:

$$(Tu, v)_V = b(u, v) \quad \forall v \in V \qquad\qquad I_V \circ T = B$$

$$I_V : V \to V^* \text{ with } \langle I_V v, w\rangle = (v, w)_V$$

- Unique solution due to Riesz representation theorem
- Using this mapping, we obtain:

$$\|u\|_E := \sup_{v\in V} \frac{b(u,v)}{\|v\|_V} = \sup_{v\in V} \frac{(Tu,v)_V}{\|v\|_V} = \frac{(Tu,Tu)_V}{\|Tu\|_V} = \|Tu\|_V$$

$$(u,u)_E = (Tu,Tu)_V$$

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Optimal Test Spaces

Ansatz for the test space:

$$V_h^{opt} := TU_h \qquad V_h^{opt} := \text{span}\{Te_j\}$$

for given trial space $U_h = \text{span}\{e_j : \text{ lin. ind.}\}$

Since T injective, $dim(V_h^{opt}) = dim(U_h)$

Definition (Ideal PG Method)

$$b(u_h, v_h) = l(v_h) \qquad \forall v_h \in TU_h$$

Computation of T

Some considerations:

$$(\varphi_i^{opt}, v)_V = (Te_i, v)_V = b(e_i, v) \qquad \forall v \in V$$

- Solve a global variational problem for each trial function e_i!
- Operator notation

$$\varphi_i^{opt} = I_V^{-1} B e_i$$

Computation of T

Remedy:

- Approximate T by T_h:

$$(T_h u_h, \tilde{v}_h)_V = b(u_h, \tilde{v}_h) \quad \forall \tilde{v}_h \in \tilde{V}_h$$

  T_h injective on U_h

- Use discontinuous elements to avoid global system (DPG)
- $\rightarrow$ topic of next talks

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The following methods are equivalent:

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Theorem (Optimal Test Space)

Using V_h^{opt} as above, we get

$$\|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E.$$

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Theorem (Minimization Property)

u_h *minimizes* $\|l - Bz_h\|_{V^*}$.

Relations to Other Methods

By definition:

$$b(u_h, v_h) = (Tu_h, v_h)_V = \langle l, v_h \rangle_V \qquad \forall v_h \in V_h$$

By construction:

$$(Tu_h, Tw_h)_V = \langle l, Tw_h \rangle_V \qquad \forall w_h \in U_h$$

Normal Equation:

$$(T^* Tu_h, w_h)_U = \langle T^* I_V^{-1} l, w_h \rangle_U \qquad \forall w_h \in U_h$$

Relations to Other Methods

Error representation function $\epsilon = I_V^{-1}(l - Bu_h) \in V$

We have

$$\|\epsilon\|_V = \|I_V^{-1}B(u-u_h)\|_V = \|T(u-u_h)\|_V = \|u-u_h\|_E$$

Theorem (Saddlepoint Formulation)

The ideal PG method

$$b(u_h, v_h) = l(v_h) \qquad \forall v_h \in TU_h$$

is equivalent to

$$\begin{aligned} (\epsilon, v)_V + b(u_h, v) &= l(v) \qquad && \forall v \in V \\ b(w_h, \epsilon) &= 0 && \forall w_h \in U_h \end{aligned}$$

Overview

Advantages & Outlook

- Optimal error estimate
- inf-sup implies discrete inf-sup for optimal V_h
- Symmetric, positive definite system
- With DG: local systems for T
- Optimal convergence rates: h^s instead of $h^{s-\frac{1}{2}}$

Thank you for your attention!

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