A Posteriori Error Control for DPG Methods

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

Overview

- Literature
- The Error Estimator
- The Abstract Error Control
- Application to the Poisson Problem
- Numerical Examples

Main Literature

Carsten Carstensen, Leszek Demkowicz, and Jay Gopalakrishnan.(2014)
A posteriori error control for DPG methods.
SIAM J. Numer. Anal., 52(3):1335-1353, 2014.

The reference for this paper is denoted by
[C. Carstensen, L. Demkowicz,& J. Gopalakrishnan,(2014)].

Overview

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The DPG-Methode

- FEM with nonstandard test-space
- least squares-FEM in a nonstandard norm

The Problem

- X Banachspace, Y Hilbertspace
- Find $x \in X$ such that

$$b(x, y) = F(y) \quad \forall y \in Y,$$

where $F \in Y^*$, or equivalently: Find $x \in X$ such that

$$Bx = F \quad \text{in } Y^*.$$

Optimal and Practical Test Space

- $X_h \subset X$
- Optimal Test Space: $\widehat{Y_h^{opt}} = T(X_h) \subset Y$, with

$$(Tx, y)_Y = b(x, y) \quad \forall y \in Y, x \in X$$

- Practical Test Space: $Y_h^{opt} = T_h(X_h) \subset Y$, with

$$(T_h x, y_h)_Y = b(x, y_h) \quad \forall y_h \in Y_h, x \in X$$

Least Squares Finite Element Method

- $X_h \subset X$
- Ideal: Find $\widehat{x}_h \in X_h$ such that

$$\widehat{x}_h := \text{arg} \min_{\xi_h \in X_h} \|B\xi_h - F\|_{Y^*}$$

holds.

- using a test space with no continuity constraints across the elements allows to localize the Norm $\|Bx - F\|_{Y^*}$.
- Practical: Find $x_h \in X_h$ such that

$$x_h := \text{arg} \min_{\xi_h \in X_h} \|B\xi_h - F\|_{Y_h^*}$$

holds.

The Error Representation Function

- the error representation function:

$$(\varepsilon_h, y)_Y = \langle F - Bx_h, y \rangle = F(y) - b(x_h, y) \quad \forall y \in Y$$

- it holds

$$\|\varepsilon_h\|_Y = \|F - Bx_h\|_{Y^*}$$

How this error representation function is obtained?

- **locally** computed on enriched space (higher polynomial degree)

The Error Estimator

We choose our error indicators e_K as

$$\eta^2 := \|F - b(x_h, \cdot)\|_{Y^*}^2 = \|\varepsilon_h\|_Y^2 = \sum_K \underbrace{\|\varepsilon_h\|_{Y(K)}^2}_{e_K}$$

Overview

The 3 Ingredients

(H1) Boundedness

$$\|b\| := \sup_{0\neq x\in X} \sup_{0\neq y\in Y} \frac{b(x,y)}{\|x\|_X \|y\|_Y} < \infty,$$

(H2) Inf-Sup Condition

$$0 < \beta := \inf_{0\neq x\in X} \sup_{0\neq y\in Y} \frac{b(x,y)}{\|x\|_X \|y\|_Y}$$

and $\{y \in Y : b(x,y) = 0, \forall x \in X\} = \{0\}$

(H3) Fortin Operator for the Adjoint System

$$\exists \Pi : Y \mapsto Y_h$$

such that

$$b(x_h, (I - \Pi)y) = 0 \qquad \forall y \in Y, x_h \in X_h$$

Properties of the Error Estimator

Definition (Reliability)

We say an error estimator is **reliable** if there is a $C_3, C_4 > 0$ such that

$$\|x - x_h\| \leq C_3\eta + C_4\text{osc}(F)$$

Definition (Efficiency)

We say an error estimator is **efficient** if there is a $C_2 > 0$ such that

$$C_2\eta \leq \|x - x_h\|$$

An Abstract Theorem

Theorem (Reliability & Efficiency) [C. Carstensen, L. Demkowicz,& J. Gopalakrishnan,(2014)]

Let (H1),(H2),(H3) be fulfilled. Let $F \in Y^*$, $x := B^{-1}F$ and an arbitrary $\widetilde{x}_h \in X_h$. Then for the error estimator $\widetilde{\eta} := \|F - B\widetilde{x}_h\|$ and the data approximation error $\text{osc}(F) := \|F \circ (I - \Pi)\|$ holds:

$$\beta^2\|x - \widetilde{x}_h\|_X^2 \leq \widetilde{\eta}^2 + (\|\Pi\|\widetilde{\eta} + \text{osc}(F))^2$$
$$\widetilde{\eta} \leq \|b\|\|x - \widetilde{x}_h\|_X$$
$$\text{osc}(F) \leq \|b\|\|I - \Pi\| \min_{\xi_h \in X_h} \|x - \xi_h\|_X$$

Proof.

Blackboard □

Some Remarks

- (H1)+(H2) $\iff$ well-posedness and $\|B^{-1}\| = \frac{1}{\beta}$
- (H3) $\iff$ stability
- constants in error bounds: $C_2 = \frac{1}{\|b\|}$, $C_3 = \frac{(1+\|\Pi\|^2)^{\frac{1}{2}}}{\beta}$, $C_4 = \frac{1}{\beta}$
- an alternative proof of best approximation property

$$\beta\|x - \widetilde{x}_h\|_X \leq (C_3 + C_4\|b\|\|I - \Pi\|) \min_{\xi_h \in X_h} \|x - \xi_h\|_X$$

- the data approximation error $\mathrm{osc}(F)$ is usually of higher order.

Overview

The Poisson Problem

Find u such that

$$-\Delta u = f \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega.$$

The Poisson Problem

Find u such that

$$\begin{aligned} -\,div(\nabla u) &= f && \text{in } \Omega, \\ \sigma + \nabla u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The Poisson Problem

Find u such that

$$\begin{aligned} div(\sigma) &= f && \text{in } \Omega, \\ \sigma + \nabla u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The Poisson Problem

Find $(\sigma, u, \hat{u}, \hat{\sigma}_n) \in X$ such that

$$\begin{aligned} -(\sigma, \nabla v)_{\Omega_h} + (\hat{\sigma}_n, v)_{\partial\Omega_h} &= (f, v)_\Omega \qquad \forall v \in H^1(\Omega_h), \\ (\sigma, \tau)_\Omega - (u, div(\tau))_{\Omega_h} + (\hat{u}, \tau.n)_{\partial\Omega_h} &= 0 \qquad \forall \tau \in H(div, \Omega_h), \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $X := L^2(\Omega) \times L^2(\Omega) \times H_0^{1/2}(\partial\Omega_h) \times H^{-1/2}(\partial\Omega_h)$.

The Poisson Problem

This finally leads to: Find $(\sigma, u, \hat{u}, \hat{\sigma}_n) \in X$ such that

$$b((\sigma, u, \hat{u}, \hat{\sigma}_n), (v, \tau)) = F(v, \tau) \quad \forall (\tau, v) \in Y$$

where $X := L^2(\Omega) \times L^2(\Omega) \times H_0^{1/2}(\partial\Omega_h) \times H^{-1/2}(\partial\Omega_h)$ and $Y := H(div, \Omega_h) \times H^1(\Omega_h)$.

The Discrete Spaces

$$X_h := \{(\sigma, u, \hat{u}, \hat{\sigma}_n) \in X : \forall K \in \Omega_h, \sigma_{|K} \in \mathbb{P}_p(K, \mathbb{R}^N),$$
$$u_{|K} \in \mathbb{P}_p(K, \mathbb{R}),$$
$$\hat{u}_{|\partial K} \in \tilde{\mathbb{P}}_{p+1}(\partial K, \mathbb{R}),$$
$$\hat{\sigma}_{n|\partial K} \in \mathbb{P}_p(\partial K, \mathbb{R})\}$$

$$Y_h := \{(\tau, v) \in Y : \forall K \in \Omega_h, \tau_{|K} \in R_{p+1}(K, \mathbb{R}),$$
$$v_{|K} \in \mathbb{P}_{p+N}(K, \mathbb{R})\}$$

Theorem

For the Poisson equation holds for this choice of spaces that there exist mesh-independent constants C_1, C_2, C_3 such that the exact solution x, an approximation $\widetilde{x}_h$ with its error estimator $\widetilde{\eta} := \|F - B\widetilde{x}_h\|_{Y^*}$ satisfy

$$C_1 \|x - \widetilde{x}_h\|_X^2 - C_2 \text{osc}(F)^2 \leq \widetilde{\eta}^2 \leq C_3 \|x - \widetilde{x}_h\|_X^2.$$

If $p \geq 1$ it additionally holds

$$\text{osc}(F)^2 \leq C_4 \sum_{K \in \Omega_h} \|h_K (f - f_K)\|_{L^2(K)}^2.$$

Hint for Proof.

Proof (H1),(H2),(H3) and apply the abstract Theorem.
(Seminar 6 $\implies$ (H1)+(H2) and Seminar 7 $\implies$ (H3)) $\square$

Overview

The Poisson Problem

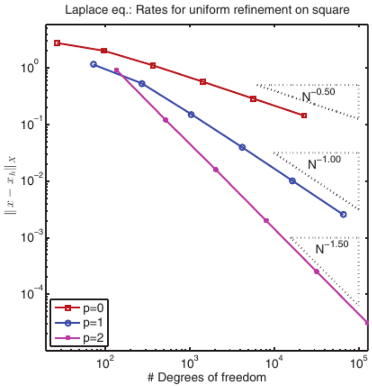
We consider the Poisson equation on an L-shaped domain. The data is given such that the exact solution is given by

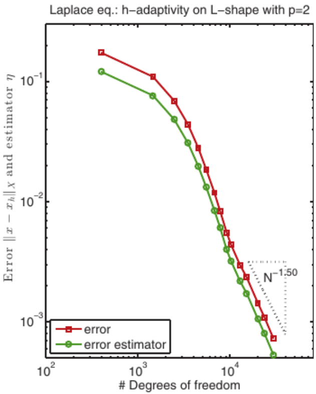
$$u(r,\phi) = r^{\frac{2}{3}}\mathsf{sin}(\frac{2}{3}(\phi + \frac{\pi}{2})),$$

and on the unit square, where the data is chosen such that the exact solution is given by

$$u(x,y) = \mathsf{sin}(\pi x)\mathsf{sin}(\pi y).$$

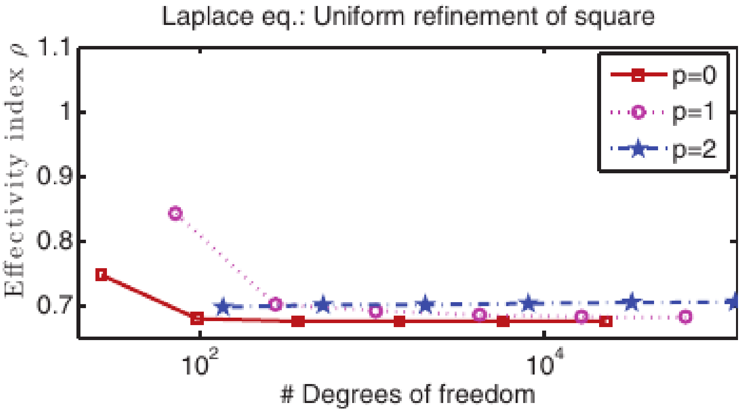
Error vs. Error Estimator





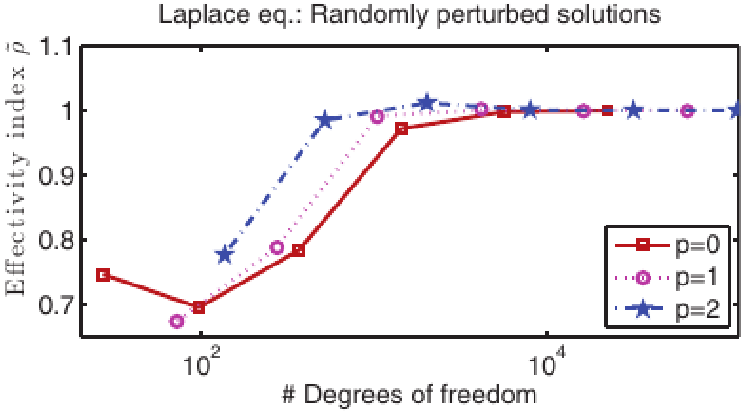
Screenshot taken form
[C. Carstensen, L. Demkowicz,& J. Gopalakrishnan,(2014)].

Effictivity Index ρ for h-refinement



Screenshot taken form
[C. Carstensen, L. Demkowicz,& J. Gopalakrishnan,(2014)].

Effictivity Index ρ for perturbed solution



Screenshot taken form
[C. Carstensen, L. Demkowicz,& J. Gopalakrishnan,(2014)].

Thanks for your attention