A class of discontinuous Petrov-Galerkin methods. III: Adaptivity

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Computational Mathematics
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Overview

Main Literature

Leszek Demkowicz, Jay Gopalakrishnan, and Antti H. Niemi (2012)
A class of discontinuous Petrov-Galerkin methods. III: Adaptivity.
Appl. Numer. Math., 62(4):396-427, 2012.

The reference for this paper is denoted by
[L. Demkowicz, J. Gopalakrishnan, & A NiemiR (2012)].

Overview

The Problem

- U, V Hilbertspaces
- Find $u \in U$ such that

$$b(u,v) = l(v) \quad \forall v \in V,$$

  where $l \in V'$.
- $b(u,v) \leq M\|u\|_U \|v\|_V \quad \forall u \in U, \forall v \in V$
- $\inf_{u\in U} \sup_{v\in V} b(u,v) \leq \gamma$

Optimal Test Space and Functions

- $U_{hp} = \text{span}\{e_j\}_{j=1}^{N_{hp}}$
- Optimal Test Functions: $\widetilde{e_j} \in V$

$$(\widetilde{e_j}, v)_V = b(e_j, v) \quad \forall v \in V$$

- $V_{hp} = \text{span}\{\widetilde{e_j}\}_{j=1}^{N_{hp}}$

The Error Representation Function

- the error function e_{hp} is given by $e_{hp} = u - u_{hp}$
- the error representation function:

$$(v_{e_hp}, v)_V = l(v) - b(u_{hp}, v) \quad \forall v \in V$$

- it holds:

$$\|v_{e_{hp}}\|_V = \|e_{hp}\|_E = \|u - u_{hp}\|_E$$

How this error representation function is obtained?

- enriched space (higher polynomial degree)

The Error Estimator

We choose our error indicators e_K as

$$\|e_{hp}\|_E^2 = \|v_{e_{hp}}\|_V^2 = \sum_K \underbrace{\|v_{e_{hp}}\|_{V(K)}^2}_{e_K}$$

Overview

- Literature
- The Energy Error Estimator
- Convection Dominated Diffusion Problem in 1D
  1. Norm of the Test Functions
  2. Global Continuity of the Error Function
  3. ϵ-Robustness of the DPG-Energynorm
- Numerical Examples
  1. Convection Dominated Diffusion Problem 1D
- Conclusions

Convection Dominated Diffusion Problem in 1D (Strong From)

Find u, σ such that

$$
\begin{aligned}
\frac{1}{\epsilon}\sigma - u' &= 0, \\
-\sigma' + u' &= f, \\
u(0) = u_0, \quad u(1) &= 0,
\end{aligned}
$$

for a given right hand side $f(x)$ on the domain $(0, 1)$.

Discretization

Let us now decompose the domain $(0,1)$ into an arbitrary partition of N elements (x_{k-1}, x_k). Now we want to find $u_k, \sigma_k, \hat{u}, \hat{\sigma}$ such that

$$\int_{x_{k-1}}^{x_k} \frac{1}{\epsilon}\sigma_k \tau + u_k \tau' dx - (\hat{u}\tau)|_{x_{k-1}}^{x_k} = 0,$$

$$\int_{x_{k-1}}^{x_k} \sigma_k \nu' - u_k \nu' dx - (\hat{\sigma}\nu)|_{x_{k-1}}^{x_k} + (\hat{u}\nu)|_{x_{k-1}}^{x_k} = \int_{x_{k-1}}^{x_k} f\nu dx,$$

for all (τ, ν) optimal test functions.

- σ_k, u_k polynomials on the $k-th$ element
- $\hat{\sigma}(x_k), \hat{u}(x_k)$: the fluxes
- $\hat{u}(0), \hat{u}(1)$ are known and moved to the right hand side

Overview

Norm of the Test Functions

The norm is defined element wise by

$$\|(\tau,\nu)\| := (\sum_{k=1}^{N} \|\tau_k\|_k^2 + \|\nu_k\|_k^2)^{\frac{1}{2}}.$$

For the local norms $\|\cdot\|_k$ we investigate two different norms:

- A weighted H^1-norm

$$\|v\|_k^2 := \int_{x_{k-1}}^{x_k} \alpha(v^2 + (v')^2)dx$$

- A mesh dependent norm

$$\|v\|_k^2 := \int_{x_{k-1}}^{x_k} \alpha(v')^2dx + \beta_k|v(x_k)|^2$$

Overview

Let $\mathcal{U} := (\sigma, u, \hat{\sigma}, \hat{u})$ be the exact solution with

- $\sigma = (\sigma_1, \cdots, \sigma_N)$.
- $u = (u_1, \cdots, u_N)$.
- $\hat{\sigma} = (\sigma(x_0), \cdots, \sigma(x_N)$.
- $\hat{u} = (u(x_0), \cdots, u(x_N))$.

and $\mathcal{U}_{hp}$ be the solution of our discretization.
Then we define

$$\mathcal{E}_{hp} := \mathcal{U} - \mathcal{U}_{hp}.$$

The Error Representation Function

The error representation function is the solution of the problem:
Find $(\phi, \psi) \in V$ such that

$$((\phi, \psi), (\delta\tau, \delta\nu))_V = b(\mathcal{E}_{hp}, ((\delta\tau, \delta\nu)) \quad \forall(\delta\tau, \delta\nu) \in V,$$

where $(\cdot, \cdot)_V$ is the inner product defined by our norm.

Test with an optimal testfunkion

- $(\tau_{\hat{\sigma}_k}, \nu_{\hat{\sigma}_k})$: optimal test function for $\hat{\sigma}_k := \hat{\sigma}(x_k)$
- $b(\mathcal{E}_{hp}, (\tau_{\hat{\sigma}_k}, \nu_{\hat{\sigma}_k})) = 0$
- $b(\mathcal{E}_{hp}, (\tau_{\hat{\sigma}_k}, \nu_{\hat{\sigma}_k})) = b_k(\mathcal{E}_{hp}, (\tau_{\hat{\sigma}_k}, \nu_{\hat{\sigma}_k})) + b_{k+1}(\mathcal{E}_{hp}, (\tau_{\hat{\sigma}_k}, \nu_{\hat{\sigma}_k}))$

The Elementwise Error Representation Function

The element wise error representation function is the solution of the problem:
Find $(\phi_k, \psi_k) \in V((x_{k-1}, x_k))$ such that

$$((\phi_k, \psi_k), (\delta\phi, \delta\psi))_V = b_k(\mathcal{E}_{hp}, (\delta\phi, \delta\psi)) \quad \forall (\delta\phi, \delta\psi) \in V((x_{k-1}, x_k)),$$

where $(\cdot, \cdot)_V$ is the inner product defined by our norm.

Global Continuity

Theorem: Continuity

The Error Representation Function is continuous

Proof: Blackboard

Theorem: Mesh-independence

For the weigthed H_1-norm the DPG energy norm of the FE-error coincides with the spectral energy norm.

Proof: Blackboard

$\Longrightarrow$ energy-norm of the error can not increase for any mesh and h and p -refinement (neglecting round off and integration errors)

Overview

Theorem: An Explicit Representation Formula

If we choose the mesh dependent norm for the test space with $\beta = 1$ on just one element then the the explicit formula is given by

$$\begin{aligned} \|(\sigma, u, \hat{\sigma}(0), \hat{\sigma}(1)\|_E^2 &= \| \int_0^x \frac{1}{\epsilon}\sigma(s)ds - u(x)\|_{\frac{1}{\alpha}}^2 \\ &+ \| -\sigma(x) + u(x) + \hat{\sigma}(0)\|_{\frac{1}{\alpha}}^2 \\ &+ |\int_0^1 \frac{1}{\epsilon}\sigma(s)ds|^2 + |\sigma(0) - \sigma(1)|^2 \end{aligned}$$

Proof: Special case of the (multi-element) energy norm (see [L. Demkowicz, J. Gopalakrishnan, & A NiemiR (2012)])

Theorem: Boundedness of the DPG(Multi-Element) Energynorm

If we choose the mesh dependent norm for the test space with $\beta_k = h_k := x_k - x_{k-1}$ it holds

$$\|(\sigma, u, \hat{\sigma}(0), \hat{\sigma}(1)\|_E^2 \leq \|(\sigma, u, \hat{\sigma}, \hat{\sigma})\|_E^2$$

Proof: (see [L. Demkowicz, J. Gopalakrishnan, & A NiemiR (2012)])

Theorem: L^2-stability for u and σ

If we choose the mesh dependent norm for the test space with $\beta = 1$ and if $\alpha(s)$ is choosen as

$$\alpha(s) := \begin{cases} \frac{\epsilon}{2} & \forall s \in (0, -\frac{\epsilon}{2}\ln(\frac{\epsilon}{2})) \\ 1 & else \end{cases}$$

$$\max\{\|u\|_{L^2}, \|u\|_{L^2}\} \leq \|(\sigma, u, \hat{\sigma}(0), \hat{\sigma}(1)\|_E^2$$

Sketch of the Proof on Black board!

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The Problem

The data in
[L. Demkowicz, J. Gopalakrishnan, & A NiemiR (2012)] was chosen such that the exact solution is given by

$$u(x) := \frac{1}{1 - e^{-\frac{1}{\epsilon}}}(1 - e^{\frac{x-1}{\epsilon}}).$$

The enriched space has polynomial degree $p + 4$ with initial $p = 0$ with

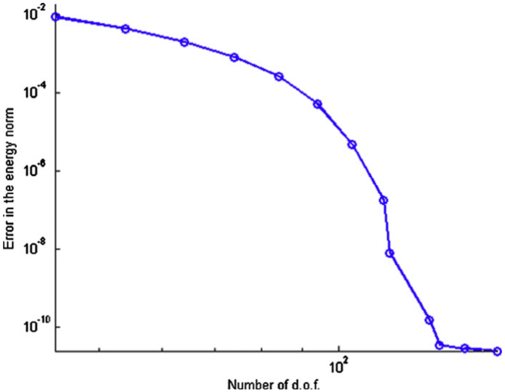
$$\alpha(x) := \begin{cases} 0.1 & \forall x \in (0, 0.25) \\ 1 & else \end{cases}.$$

The Poor Man Greed hp-Algorithm

```
Set δ = 0.5
do while δ > 0.1
  solve the problem on the current mesh
  for each element K in the mesh
    compute element error contribution e_K
  end of loop through elements
  for each element K in the mesh
    if e_K > δ^2 max_K e_K then
      if new h ⩾ ϵ then
        h-refine the element
      elseif new p ⩽ p_max then
        p-refine the element
      endif
    endif
  end of loop through elements
if (new N_dof = old N_dof) reset δ = δ/2
end of loop through mesh refinements
```

Screenshot taken form
[L. Demkowicz, J. Gopalakrishnan, & A NiemiR (2012)].

$\epsilon = 10^{-3}$



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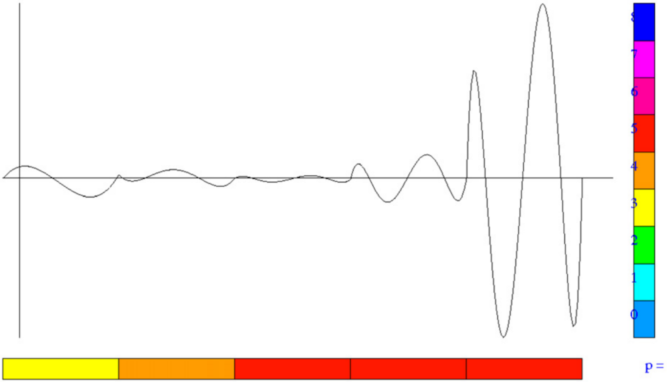

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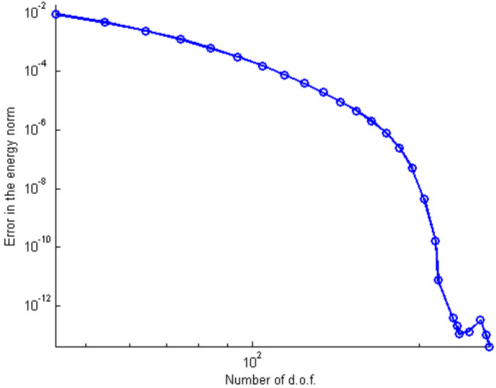
$\epsilon = 10^{-6}$



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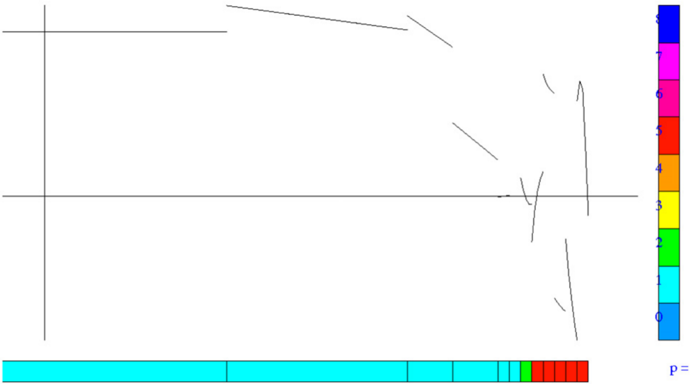

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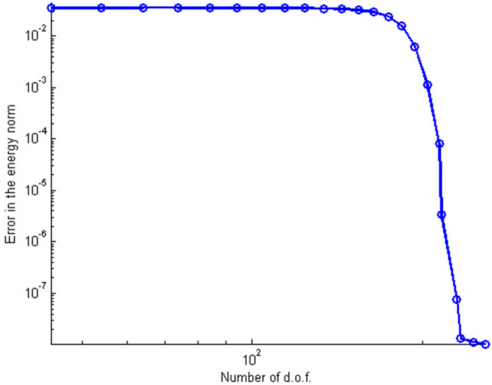
$\epsilon = 10^{-6}$ with correct α



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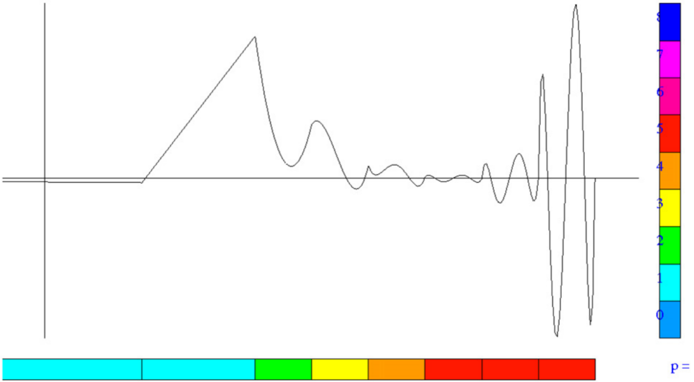

Screenshot taken form
[L. Demkowicz, J. Gopalakrishnan, & A NiemiR (2012)].

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The discrete Problem in 2D

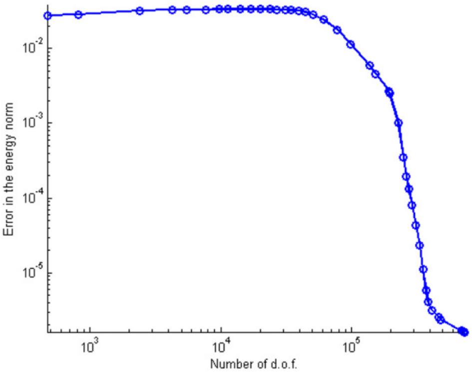
Now we want to find $u_K, \sigma_K, \hat{u}, \hat{\sigma}$ such that

$$\int_K \frac{1}{\epsilon}\sigma_K \tau + u_K div(\tau) dx - \int_{\partial K} (\hat{u}\tau.n) ds_x = 0,$$

$$\int_K \sigma_K \nabla\nu - u_K \beta \nabla\nu dx - \int_{\partial K} (\hat{\sigma}.n\nu) - (\hat{u}\beta.n\nu) ds_x = \int_K f\nu dx,$$

for all (τ, ν) optimal test functions. Here we use a weighted H^1-norm for ν and and a weighted $H(div)$ norm for τ.

$\epsilon = 10^{-7}$ with $\beta = (1, 1)$



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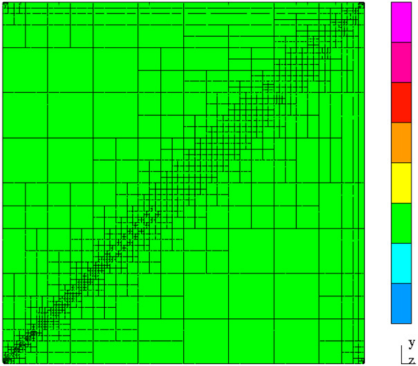
$\epsilon = 10^{-7}$ with $\beta = (1, 1)$



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Conclusions

- error estimator in the energy norm
- continuity of the error representation functions.
- construction of a mesh dependent norm, which grants stability in ϵ.

Thanks for your attention