Discontinuous-Petrov-Galerkin for pure Convection

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Overview

Overview

- Almost optimal test functions for pure convection
- Locally computed test functions
- Trial space: Discontinuous polynomials & fluxes

Recap

- Energy Norm: $\|u\|_E := \sup_{\|v\|_V=1} b(u,v)$
- $T : U \longrightarrow V, \quad (Tu, v)_V = b(u,v) \quad \forall v \in V$
- Optimal Test Space: $V_n = T(U_n)$
- Resulting solution u_n is optimal:

$$\|u - u_n\|_E = \inf_{w_n \in U_n} \|u - w_n\|_E$$

4 Steps guideline

4 steps are needed to obtain a DPG-method:

1. Mesh dependent variational formulation with interelement discontinuities
2. Accurate trial spaces
3. Approximate optimal test functions per element:
   - Approximate T: $(T_n u_n, \tilde{v}_n)_V = b(u_n, \tilde{v}_n) \qquad \forall \tilde{v}_n \in \tilde{V}_n$
   - $T_n : U_n \longrightarrow \tilde{V}_n$ is injective
   - $V_n = T_n(U_n)$;
4. Solve a symmetric positive definite system.

Problem

Pure Convection or Transport-Equation:

$$\begin{aligned} \beta\nabla u &= f \quad in\ \Omega \\ u &= u_0 \quad on\ \Gamma_{in} := \{x \in \partial\Omega | \beta(x)\cdot n(x) < 0\} \end{aligned}$$

Mesh dependent variational formulation:
Find $u \in L^2(\Omega), q \in L^2(\Gamma_h)$:

$$b((u,q),v) = l(v) \qquad \forall v \in H_\beta(K) := \{v \in L^2 | \beta\nabla v \in L^2\}$$

$$b((u,q),v) := \sum_K \int_K -u(\beta\cdot\nabla)v\ dx + \int_{\partial K\setminus\Gamma_{in}} sgn(\beta\cdot n)qv\ ds$$

$$l(v) = \sum_K \int_K fv\ dx + \int_{\partial K\cap\Gamma_{in}} (\beta\cdot n)u_0 v\ ds$$

The flux is introduced as $q := |\beta\cdot n|u$ and serves as sole coupling component.

Finite Trial Spaces

The construction of trial spaces reflects the concept of optimal test spaces.

- u
  - Accurate approximation by polynomials
  - Corresponding test functions restricted on the same element
- q
  - Simple approximation (scalars for 1D) per face
  - Corresponding test functions on two neighboring elements

The test functions in $H_\beta(K)$ are also discontinuous over the elements.

Model problem 1D: setting

Let

- $\beta = 1$
- $K = (x_1, x_2)$,

then $H_\beta(K) = H^1((x_1, x_2))$.
To obtain a Hilbertspace, we choose the inner product:

$$(v, w)_V = \int_{x_1}^{x_2} v' w' dx + v(x_2) w(x_2)$$

The bilinear form of the Problem is given by

$$b((u, q), v) = -\int_{x_1}^{x_2} u v' dx + q v(x_2).$$

Model problem 1D: trialspace

The next step is to select a space of trialfunctions for (u, q):

$$U_p = \mathbb{P}_p(K) \times \mathbb{R}$$

where $\mathbb{P}_p$ denotes the polynomials on K of degree less or equal p

Model problem 1D: optimal testspace

The optimal testfunctions $v_{u,q} := T(u,q)$ must satisfy:

$$\left(v_{(u,q)}, w\right)_V = b((u,q), w) \quad \forall w \in H^1(\Omega)$$

i.e. for $v_q = T(0,q)$

$$\int_{x_1}^{x_2} v_q' w' dx + v_q(x_2) w(x_2) = w(x_2)$$

respectively for $v_u = T(u,0)$

$$\int_{x_1}^{x_2} v_u' w' dx + v_u(x_2) w(x_2) = -\int_{x_1}^{x_2} u w' dx$$

Model problem 1D: optimal testspace

The optimal testfunctions are

$$v_{(0,1)}(x) = 1$$

$$v_{(u,0)}(x) = \int_x^{x_2} u(s)ds.$$

Model problem 1D multielement: setting

Given:

- $\beta = 1$
- $\Omega = (x_1, x_n)$ split into elements (x_i, x_{i+1}),

Chosen inner product:

$$(v, w)_V = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} v' w' dx + \alpha_i v^{up}(x_i) w^{up}(x_i)$$

$v^{up}(x_i) = v(x_i - 0)$ denotes the limit from the left.
$v^{dn}(x_i) = v(x_i + 0)$ denotes the limit from the right.
The bilinear form is:

$$b((u, q), v) = \sum_{i=1}^{n} \left[- \int_{x_{i-1}}^{x_i} u v' dx + q_i v^{up}(x_i) - q_{i-1} v^{dn}(x_{i-1}) \right]$$

Model problem 1D multielement: trialspace

The trialspace is set to

$$U_h := \{(w_h, q_1, \cdots, q_n) : w_h|_{(x_{i-1}, x_i)} \in \mathbb{P}^p, q_i \in \mathbb{R}\}$$

with the understanding that $q_0 = 0$.

Model problem 1D multielement: optimal testfunctions

As we allowed discontinuity between the elements , the optimal testfunction can be computed seperately for each element. Therefore we can use the result of the one elemental case to obtain for $w \in \mathbb{P}^p(x_i, x_{i+1})$:

$$v_{(w,0,\cdots,0)}(x) = \int_x^{x_{i+1}} w(s)ds$$

Model problem 1D multielement: optimal testfunctions

For the optimal testfunction v_i corresponding to the unit flux at x_i, it is a bit more tricky, it has to fullfill

$$\int_{x_{i-1}}^{x_i} v_i' w' \, dx + \alpha_i v_i^{up}(x_i) w^{up}(x_i) = w^{up}(x_i)$$

for all $w \in H^1(x_{i-1}, x_i)$ and

$$\int_{x_i}^{x_{i+1}} v_i' w' \, dx + \alpha_{i+1} v_i^{up}(x_{i+1}) w^{up}(x_{i+1}) = -w^{dn}(x_i)$$

for all $w \in H^1(x_i, x_{i+1})$.

Model problem 1D multielement: optimal testfunctions

And this yields that

$$v_i(x) = \begin{cases} \frac{1}{\alpha_i} & \text{for } x \in (x_{i-1}, x_i) \\ x - \frac{1+\alpha_{i+1} x_{i+1}}{\alpha_{i+1}} & \text{for } x \in (x_i, x_{i+1}) \\ 0 & \textit{elsewhere} \end{cases}$$

Model problem 1D multielement:statements

Theorem

The energy norm is given by

$$||(u, q_1, \cdots, q_n)||_E^2 = \sum_{i=1}^{n} \frac{|q_i - q_{i-1}|^2}{\alpha_i} + \int_{x_{i-1}}^{x_i} |u - q_{i-1}|^2.$$

Model problem 1D multielement:statements

Theorem

For all $u \in L^2(x_0, x_n)$ *and all* $q = (q_1, \cdots, q_n) \in \mathbb{R}^n$*, the inf sup conditions*

$$||u||^2 + ||q||_h^2 \leq \gamma ||(u, q_1, \cdots, q_n)||_E^2$$

holds, where $||u||$ *denotes the* $L^2(x_0, x_n)$ *norm,*
$||q||_h = \sum_{i=1}^{n} |q_{i-1}|(x_i - x_{i-1})$*, and* $\gamma = max(3\kappa, 2)$*, with*
$\kappa = \sum_{l=1}^{n} \sum_{j=1}^{l-1} \alpha_j (x_l - x_{l-1})$.

Model problem 1D multielement:statements

Theorem

The test space can be characterized as

$$V_h = \{v : v|_K \in \mathbb{P}^{p+1} \textit{for all elements } K\}$$

Model problem 1D multielement:statements

Theorem

The solution $(u_h, q_{h,1} \cdots , q_{h,n})$ of this DPG is independent of $\{\alpha_i\}$.

Model problem 1D multielement:statements

Theorem

The error in the fluxes $q_{h,i}$ is zero, i.e. $q_{h,i} = q_i$.

Model problem 1D multielement:statements

Theorem

The solution u_h equals the L^2 projection of the exact solution.

2D DPG-A Assumptions

For the 2D problem we need some assumptions:

- $div\beta = 0$
- Babuška-Aziz is fulfilled
  implies that β has no closed loops
- Initial mesh is sufficiently fine

Inner product:

$$(v, \delta_v)_V := \sum_K \int_K \partial_\beta v \partial_\beta \delta_v \ dx + \int_K v\delta_v \ dx$$

The trial space is chosen as:

$$w_h|_K \in P^p(K)$$
$$\phi_h|_E \in P^{p+1}(E)$$

Optimal Test Function

We need to find test functions $v \in P^{p+2}$ for a basis of the trial space:

$$\int_K \partial_\beta v \partial_\beta \delta_v + v\delta_v \ dx = \int_K u \partial_\beta \delta_v \ dx + \int_{\partial K} sgn(\beta \cdot n) q \delta_v \ ds$$

$\delta_v \in H_\beta(K)$ can be impractical, instead we choose $\delta_v|_K \in P^{p+2}(K)$ as local approximation

The linear system can consequently be solved by standard methods.

Streamline Coordinates

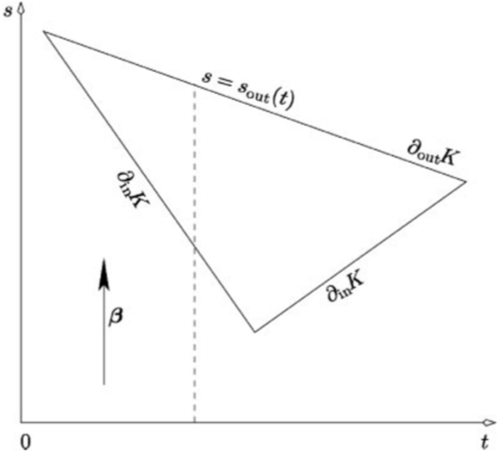


FIG. 1. Streamline crossing an element.

2D DPG-X

Solve the equation for the optimal test function analytically.
Different inner product:

$$(v, \delta_v)_V := \int_K \partial_\beta v \partial_\beta \delta_v \; dx + \int_{\partial_{out} K} v \delta_v \; dx$$

- v_u is a polynomial with degree $p + 1$ along the streamline coordinate and degree p in some second coordinate.
- $v_{\phi_{out}}$ is the constant extension along streamlines
- $v_{\phi_{in}}$ is a linear function along streamlines

When viewed on a streamline the results are the same as in 1D.
Introduces discontinuities
Only implemented for constant β, possibly approximations for general case

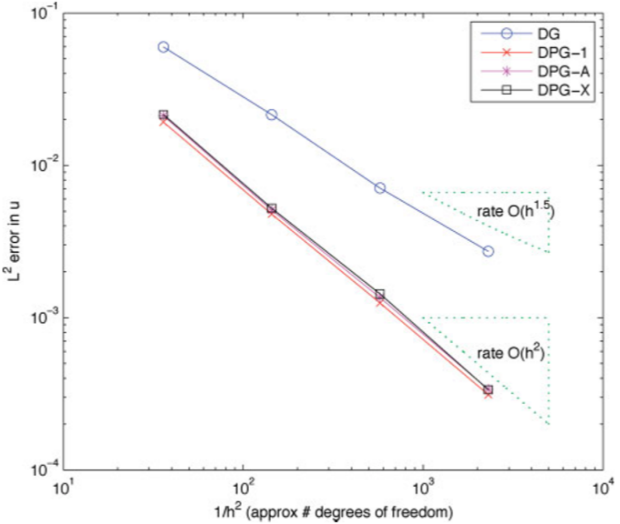
Summary

DPG-1 is the method from their first paper

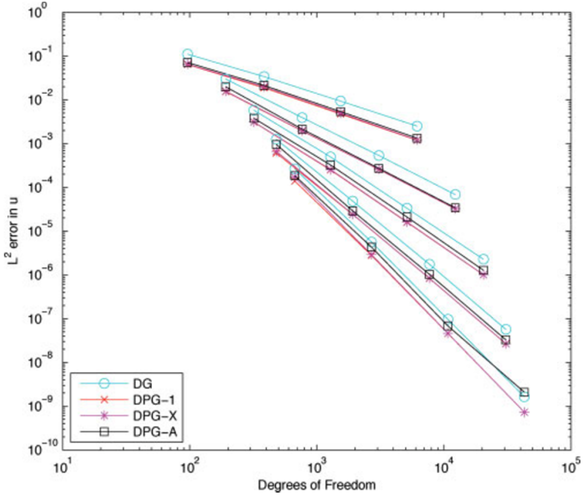
We have the following results:

- DPG-1 and DPG-X have similar test spaces
- DPG-X and DPG-A result in pos. def. symmetric matrices
- DPG-A is the easiest to implement
- DPG-A and DPG-X can be extended to the convection-dominated diffusion case

Numerical results h-refinement



Numerical results hp-refinement



$p = 1, 2, 3, 4, 5$; one line for each p

Literature

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