# Seminar Numerische Mathematik
# Discontinuous Petrov-Galerkin Methods

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## 1 Introduction

### 1.1 Variational formulations

- Standard variational problems: Find $u \in V$ such that

$$b(u,v) = \ell(v) \quad \text{for all } v \in V, \tag{1}$$

  where $V$ is a Hilbert space, $b$ is a bilinear form on $V \times V$ and $\ell$ is a linear functional on $V$.

  Typical application: diffusion problems

$$\begin{aligned} -\operatorname{div}(\alpha \nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- More general variational problems: Find $u \in U$ such that

$$b(u,v) = \ell(v) \quad \text{for all } v \in V, \tag{2}$$

  where $U$ and $V$ are Hilbert spaces, $b$ is a bilinear form on $U \times V$ and $\ell$ is a linear functional on $V$.

  Typical application: convection-dominated diffusion

$$\begin{aligned} -\varepsilon\, \Delta u + \beta \cdot \nabla u &= f \quad \text{in } \Omega, \\ u &= u_0 \quad \text{on } \partial\Omega, \end{aligned}$$

  where $\varepsilon$ is a small positive number.

  Limit case $\varepsilon \to 0$: pure convection:

$$\begin{aligned} \beta \cdot \nabla u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma_{\text{in}}, \end{aligned}$$

  where $\Gamma_{\text{in}} = \{x \in \partial\Omega \colon \beta \cdot n(x) < 0\}$.

## 1.2 Discretization methods

- Galerkin methods for (1): Find $u_h \in V_h$ such that

$$b(u_h, v_h) = \ell(v_h) \quad \text{for all } v_h \in V_h$$

with

$$V_h \subset V.$$

Typical choice: $V_h$ is a finite element space

- Petrov-Galerkin methods for (2): Find $u_h \in U_h$ such that

$$b(u_h, v_h) = \ell(v_h) \quad \text{for all } v_h \in V_h$$

with

$$U_h \subset U_h \quad \text{and} \quad V_h \subset V.$$

Typical choice: $U_h$ and $V_h$ are finite element spaces

Functions in $U_h$ and $V_h$ are called trial functions and test functions, respectively.

## 1.3 Error analysis

- Natural inner product for the case $U = V$ and $b$ symmetric and coercive:

$$(u, v)_E = b(u, v)$$

with associated norm (**energy norm**).

$$\|u\|_E = b(u, u)^{1/2}.$$

Consequence for Galerkin methods: Optimal error estimates

$$\|u - u_h\|_E = \inf_{w_h \in V_h} \|u - w_h\|_E.$$

- Extension to the general case:

$$\|u\|_E = \sup_{0 \neq v \in V} \frac{b(u, v)}{\|v\|_V}$$

If $\|u\|_E$ is a norm, it is again called the **energy norm**.

Observe that, for the case $U = V$, $b$ symmetric and coercive, and $\|v\|_V = b(v, v)^{1/2}$, we have

$$\sup_{0 \neq v \in V} \frac{b(u, v)}{\|v\|_V} = \sup_{0 \neq v \in V} \frac{(u, v)_V}{\|v\|_V} = \|u\|_V = b(u, u)^{1/2}$$

**Question:** for what choice of $V_h$ do we obtain similar optimal error estimates

$$\|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E?$$

### 1.4 Practical realization:

More general classes of discretization methods are needed.

- Discontinuous Galerkin methods:
  $$V_h \not\subset V.$$
  Typical choice: finite element spaces without interelement continuity.

- Discontinuous Petrov-Galerkin methods: in general
  $$U_h \not\subset U \quad \text{and/or} \quad V_h \not\subset V.$$
  Here we consider only the case
  $$U_h \subset U \quad \text{and} \quad V_h \not\subset V.$$
  Typical choice for $V_h$: finite element spaces without interelement continuity for the test functions.

# 2 Talks

- 2017-10-17:

  **A class of discontinuous Petrov-Galerkin (DPG) methods: optimal test functions**

  literature: small parts from [1], [2], [3], [4], [5], [6], [7]

- 2017-11-07:

  **DPG methods for pure convection in 1D and 2D**

  literature: [2], [1],

- 2017-11-14:

  **DPG methods for convection dominated diffusion in 1D and 2D**

  literature: [2]

- 2017-11-21:

  **DPG methods for time-harmonic wave propagation in 1D**

  literature: [4]

- 2017-11-28:

  **Adaptivity**

  literature: [3]

# 3 Literature

**The four basic papers:**

[1] [2] [3] [4]

**Error analysis:**

- a priori error estimates: [8] (ideal DPG), [9] (practical DPG)
- a posteriori error estimates: [5]

**Particular applications:**

- thin bodies: [10]
- robust methods for convection-dominated diffusion: [11]
- Stokes problem: [5], [12]
- wave equation (space-time): [13]

**Additional material:**

**Five Lectures on DPG Methods** by Jay Gopalakrishnan: [6]
**Plenary talk at LSSC' 2017** by Jay Gopalakrishnan: [7]

# References


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[13] J. Gopalakrishnan and P. Sepulveda. A spacetime DPG method for acoustic waves. *ArXiv e-prints*, September 2017.