Isogeometric analysis based shape optimization

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Overview

1. The model problem and short revision
2. Algorithms
   - Descent direction via auxiliary problem
   - Descent direction via kernel function
3. Numerical tests
   - Standard approach
   - Heuristic approach
   - Approach in vvRKHS

Outline

1. The model problem and short revision
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Transmission problem

- From M.Eigel and K.Sturm, *Reproducing kernel Hilbert spaces and variable metric algorithms in PDE constrained shape optimization*, 2016
-

$$\min_{\Omega} J(\Omega) = \int_D |u - u_d|^2 dx$$
$$\text{s.t.} - div(\beta_+ \nabla u) = f \text{ in } \Omega^+$$
$$-div(\beta_- \nabla u) = f \text{ in } \Omega^-$$
$$u = 0 \text{ on } \partial D$$
$$[\![u]\!] = 0, \; [\![\beta \frac{\partial u}{\partial n}]\!] = 0 \text{ on } \partial\Omega^+ \cap \partial\Omega^-$$

where $D \subset \mathbb{R}^2$ bounded, $\Omega_+ := \Omega$, $\Omega_- := D \backslash \overline{\Omega}$, $f, u_d \in H^1(D)$,
$\beta = \begin{cases} \beta_+ \text{ on } \Omega_+ \\ \beta_- \text{ on } \Omega_- \end{cases}$, $\beta_+, \beta_- > 0$

Transmission problem

- Deriving the variational formulation for the constraint yields

$$\int_D \beta_\Omega \nabla u \cdot \nabla v \; dx = \int_D fv \; dx \qquad \forall v \in H_0^1(D)$$

with $\beta_\Omega := \beta_+\chi + \beta_-(1-\chi)$, $\chi := \chi_\Omega$.

- The final problem formulation is

$$\min_\Omega \; J(\Omega) = \int_D |u - u_d|^2 dx$$

$$\text{s.t.} \quad \int_D \beta_\Omega \nabla u \cdot \nabla v \; dx = \int_D fv \; dx \qquad \forall v \in H_0^1(D)$$

Revision

Definition (Eulerian semi-derivative)

Let $D \subset \mathbb{R}^d$ open, $J : \Xi \subset \mathcal{P}(D) \to \mathbb{R}$ be a shape function defined on subsets of D. Let $\Omega \in \Xi$ and $X \in C^k(\overline{D}, \mathbb{R}^d)$, $k \geq 1$, be such that $\Phi_t(\Omega) \in \Xi$ for all $t > 0$ sufficiently small. Then the *Eulerian semi-derivative* of J at Ω in direction X is defined by

$$dJ(\Omega)(X) := \lim_{t \searrow 0} \frac{J(\Phi_t(\Omega)) - J(\Omega)}{t}$$

where $\Phi_t(\Omega)$ is the flow of X.

Revision

Definition (Shape differentiable)

Let the assumptions of the previous definition hold. Then J is said to be *shape differentiable* at Ω if for some $k \geq 1$ the Eulerian semi-derivative $dJ(\Omega)(X)$ exists for all $X \in C_0^k(\overline{D}, \mathbb{R}^d)$ and

$$X \mapsto dJ(\Omega)(X)$$

is linear and continuous on $C_0^k(\overline{D}, \mathbb{R}^d)$.

Revision

- Numerical tests with predetermined vector field
  - Without L^2 projection
  - With L^2 projection

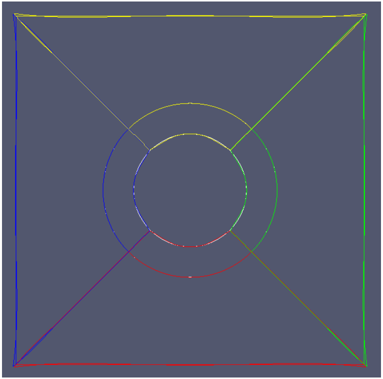

Figure: Multipatch with the displaced circle

Revision

- Numerical tests with computed vector field with simple optimal domain

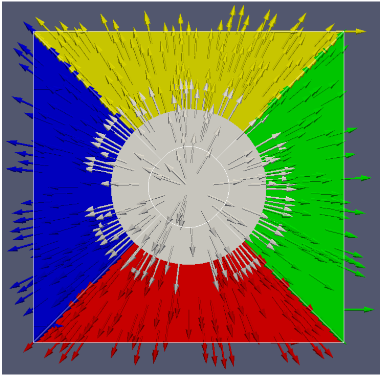

Figure: Plot of the steepest ascend, the initial shape and the optimal shape(white line)

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Shape optimization

A descent direction for the model problem can be computed in four steps:

1. Computation of u_h which solves

$$\int_D \beta_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_D f_h v_h \, dx \qquad \forall v_h \in V_h \subset H_0^1(D)$$

2. Computation of p_h which solves

$$\int_D \beta_\Omega \nabla v_h \cdot \nabla p_h \, dx = -\int_D 2(u_h - u_d) v_h \, dx \qquad \forall v_h \in V_h \subset H_0^1(D)$$

3. Computation of a descent direction X_h such that

$$dJ^h(\Omega)(X_h) < 0$$

as solution of the problem

$$(X_h, \varphi_h) = -dJ^h(\Omega)(\varphi_h) \qquad \forall \varphi_h \in W_h \subset H^1_0(D, \mathbb{R}^2)$$

where $(.,.)$ is any positive definite bilinear form on $H^1(D, \mathbb{R}^2)$

4. Move the geometry in the direction of X_h

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Reproducing kernel Hilbert spaces

Definition (Scalar reproducing kernel)

Let $\mathcal{H}$ be a Hilbert space of functions $f : \mathcal{X} \to \mathbb{R}$. A bivariate function

$$k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

is called scalar reproducing kernel for $\mathcal{H}$ if

1. $k(\cdot, x) \in \mathcal{H} \; \forall x \in \mathcal{X}$
2. $f(x) = (f, k(\cdot, x))_{\mathcal{H}(\mathcal{X})} \; \forall f \in \mathcal{H}, \forall x \in \mathcal{X}$

k is called a radial scalar kernel if there exists a function $\gamma : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathcal{X}$, $k(x, y) = \gamma(|x - y|)$

Reproducing kernel Hilbert spaces

Theorem

A Hilbert space of real valued functions $\mathcal{H}$ has a unique scalar reproducing kernel $\Leftrightarrow$ The point evaluation is a continuous and linear functional

- A scalar reproducing kernel is symmetric and positive semi-definite
- A scalar reproducing kernel for a Hilbert space of real valued functions is unique

Reproducing kernel Hilbert spaces

Definition (Matrix valued reproducing kernel)

Let $\mathcal{H}$ be a Hilbert space of functions $f : \mathcal{X} \to \mathbb{R}^d$. A bivariate function

$$K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^{d,d}$$

is called matrix valued reproducing kernel for $\mathcal{H}$ if

1. $K(x, \cdot)a \in \mathcal{H} \; \forall x \in \mathcal{X}, \forall a \in \mathbb{R}^d$
2. $a \cdot f(x) = (a \otimes \delta_x)f = (f, K(x, \cdot)a)_{\mathcal{H}(\mathcal{X},\mathbb{R}^d)} \; \forall f \in \mathcal{H}, \forall a \in \mathbb{R}^d$

K is called a radial matrix kernel if there exists a function $\gamma : \mathbb{R} \to \mathbb{R}^{d,d}$ such that for all $x, y \in \mathcal{X}$, $K(x, y) = \gamma(|x - y|)$

Reproducing kernel Hilbert spaces

Theorem

A Hilbert space of vector valued functions $\mathcal{H}$ has a unique matrix valued reproducing kernel $\Leftrightarrow$ The evaluation map

$$\begin{aligned} \mathcal{H}(\mathcal{X}, \mathbb{R}^d) &\to \mathbb{R}, \\ f &\mapsto (a \otimes \delta_x) f = f(x) \cdot a \end{aligned}$$

is continuous.

- A matrix valued reproducing kernel need not to be symmetric
- If a matrix valued reproducing kernel is symmetric it is also positiv semi-definite
- A Hilbert space of vector valued functions $\mathcal{H}$ with a reproducing matrix kernel is called (vvRKHS)

In vvRKHS we have explicit formulas for the descent direction which can be computed in four steps:

1. Computation of u_h which solves

$$\int_D \beta_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_D f_h v_h \, dx \qquad \forall v_h \in V_h \subset H_0^1(D)$$

2. Computation of p_h which solves

$$\int_D \beta_\Omega \nabla v_h \cdot \nabla p_h \, dx = -\int_D 2(u_h - u_d) v_h \, dx \qquad \forall v_h \in V_h \subset H_0^1(D)$$

Shape optimization

3. Computation of a descent direction X_h evaluated at y with

$$(X_h)(y) = \sum_{i=1}^{d} \Big(\int_D S_1(x) : \partial K_i(x,y) + S_0(x) \cdot K_i(x,y) dx - \int_{\partial D} S_1(s)\nu(s) \cdot K_i(x,y) ds \Big) e_i$$

where e_i denotes the i-th unit vector in $\mathbb{R}^d$,
$S_1 := S_1(u_h, u_d, p_h) \in L_1(D, \mathbb{R}^{d,d})$, $S_0 := S_0(u_h, u_d, p_h) \in L_1(D, \mathbb{R}^d)$

4. Move the geometry in every evaluation point in the direction of X_h

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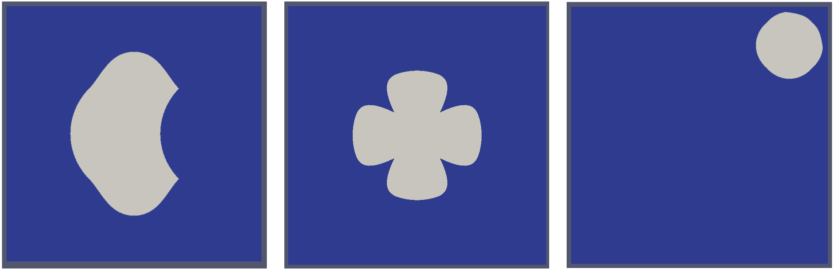
Considered optimal domains

- 3 different optimal domains



(a) (b) (c)

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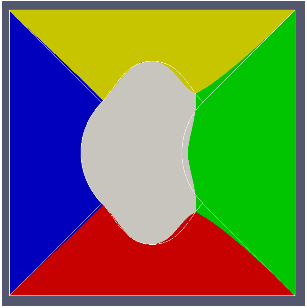
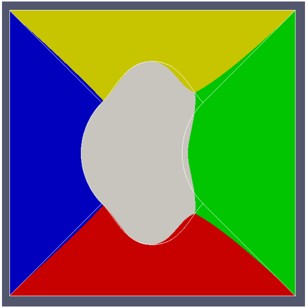
Standard approach

- Follow the previous 4 step algorithm to obtain a descent direction
- Various bilinear forms



(a) 725 iterations, Bilinearform $b(X_h, \varphi_h) = (X_h, \varphi_h)_{H^1}$, J = 1.2e-5



(b) 1000 iterations, Bilinearform $b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2} + 10(\nabla X_h, \nabla \varphi_h)_{L^2}$, J = 4.16e-6, J = 1.15e-5 after 725 iterations

Standard approach

Change of the bilinear form from
$b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2} + 10(\nabla X_h, \nabla \varphi_h)_{L^2}$ to
$b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2} + 0.1(\nabla X_h, \nabla \varphi_h)_{L^2}$ during the iteration process leads to

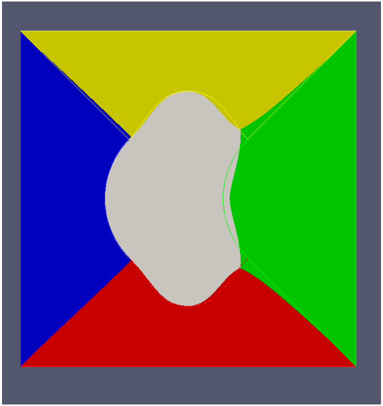
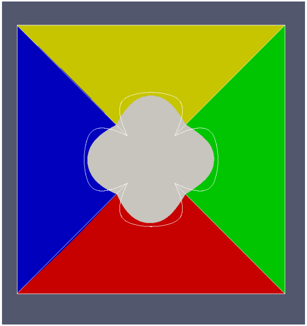
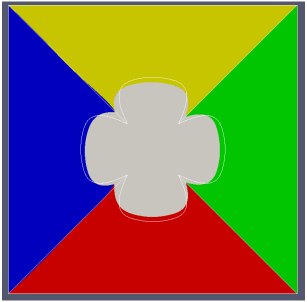


Figure: J = 3.33e-7

Standard approach

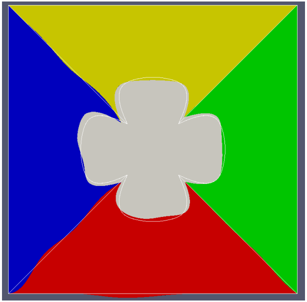


(a) Bilinearform $b(X_h, \varphi_h) = (X_h, \varphi_h)_{H^1}$, J = 0.0001904

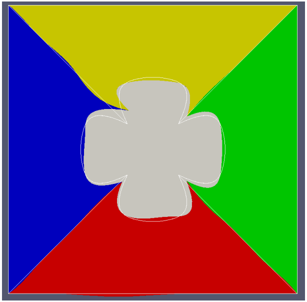


(b) Bilinearform $b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2} + 0.1(\nabla X_h, \nabla \varphi_h)_{L^2}$, J = 0.0011

Standard approach



(a) Bilinearform $b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2} + 0.001(\nabla X_h, \nabla \varphi_h)_{L^2}$, J = 0.00092



(b) Bilinearform $b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2}$, J = 0.0015

Standard approach

For the pure translation problem I got

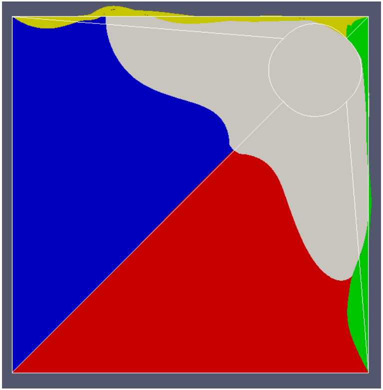


Figure: Bilinearform $b(X_h, \varphi_h) = (X_h, \varphi_h)_{H^1}$

Outline

Heuristic ansatz

- Idea from Volker Schulz and Martin Siebenborn from Trier
- The shape derivative depends only on perturbations on the interface
- Set all entries on rhs to 0 which lie not on the interface
- Then solve the auxiliary problem with the new rhs

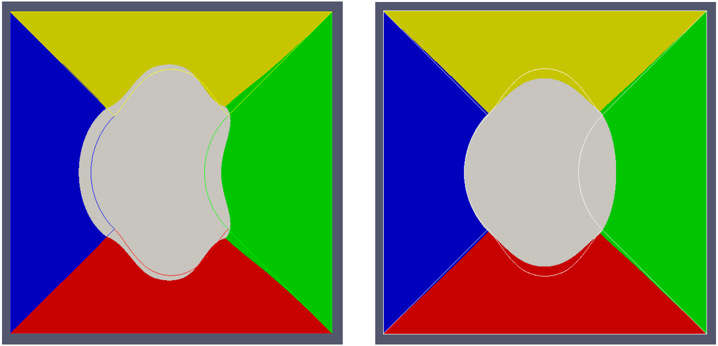
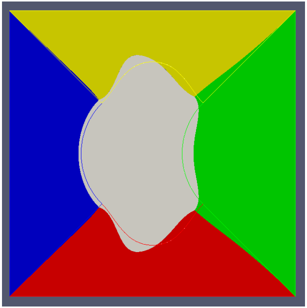
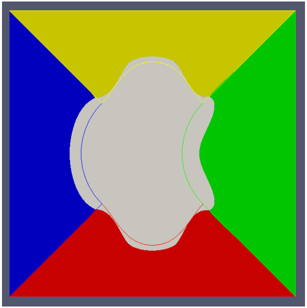


Figure: No descent in the shape functional after 3 iterations, $b(X_h, \varphi_h) = (X_h, \varphi_h)_{H^1}$, J = 0.047 (Standard: 1.2e-5)

Heuristic ansatz



(a) Bilinearform $b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2} + 10(\nabla X_h, \nabla \varphi_h)_{L^2}$, J = 0.053 (Standard: 4.16e-6)



(b) Bilinearform $b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2} + 0.1(\nabla X_h, \nabla \varphi_h)_{L^2}$, J = 0.066

- Change of the bilinear form from $b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2} + (\nabla X_h, \nabla \varphi_h)_{L^2}$ to $b(X_h, \varphi_h) = (X_h, \varphi_h)_{L^2} + 0.1(\nabla X_h, \nabla \varphi_h)_{L^2}$ during the iteration process leads to

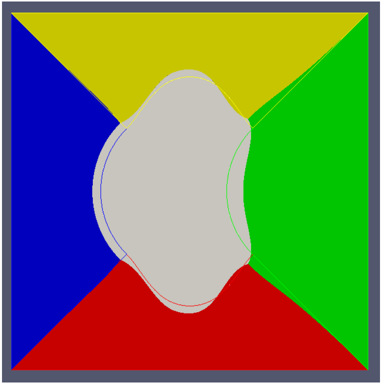


Figure: J = 0.016

Heuristic ansatz

Conclusion

- Some extra work before the auxiliary problem can be solved
- Minimum in only a few iterations
- Shape functional remains larger

$\Rightarrow$ Does not really pay off in case of IgA

Outline

Shape optimization in vvRKHS

- 2 radial kernels tested
  1. Gauss kernel

$$K(x,y) = (e^{-(|x-y|^2)/\sigma})\mathcal{I}$$

  2. C^2 Wendland kernel

$$K(x,y) = (1 - \frac{|x-y|}{\sigma})_+^4 (4\frac{|x-y|}{\sigma} + 1)\mathcal{I}$$

  where σ is a scaling factor to ensure a change of the metric during the iteration process

Shape optimization in vvRKHS

Init: $n = 0$, $\gamma > 0$, $\sigma > 0$, $N \in \mathbb{N}$, $\Omega_0 \subset D$
while n $\leq$ N do

1. compute X_h
2. decrease $t > 0$ until $J^h((id - tX_h)(\Omega_n)) < J^h(\Omega_n)$ and set $\Omega_{n+1} \leftarrow (id - tX_h)(\Omega_n)$
3. if $J^h(\Omega_n) - J^h(\Omega_{n+1}) \geq \gamma(J^h(\Omega_0) - J^h(\Omega_1))$ then

   step accepted: continue program;

   else

   decrease $\sigma \leftarrow q\sigma, q \in (0, 1)$;

   end
4. increase $n \leftarrow n + 1$;

end

Variable metric algorithm in vvRKHS

Tests with the scaled Gauss kernel

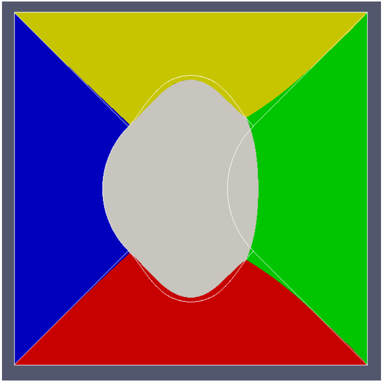
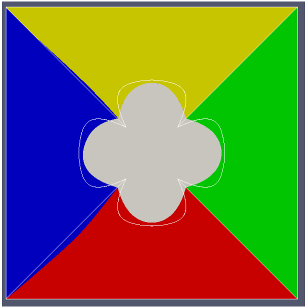
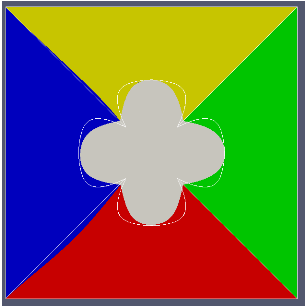


Figure: $J = 0.0060$

Shape optimization in vvRKHS

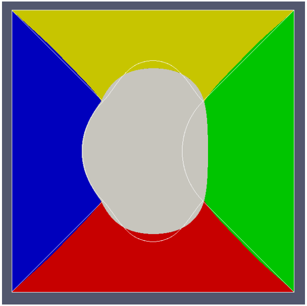


(a) J = 0.0025



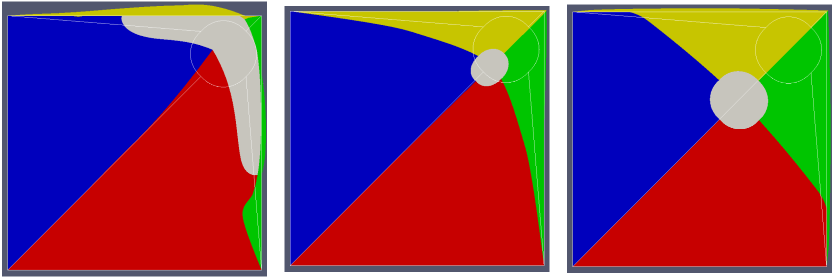
(b) J = 0.00044, 1 x uniform refined

Tests with the scaled C^2-Wendland kernel



(a) $J = 0.0067$

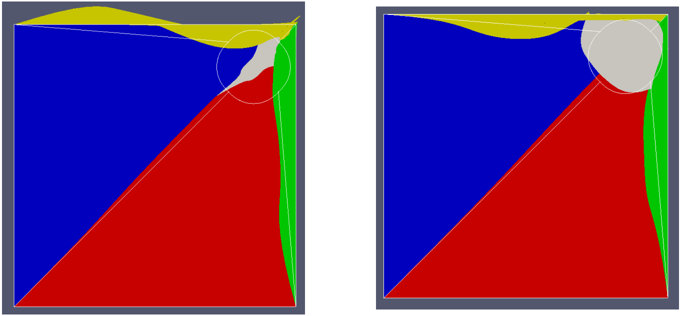
Tests to solve the pure translation problem with the Gauss kernel and different values σ



(a) $\sigma = 1$ (b) $\sigma = 10$ (c) $\sigma = 25$

Shape optimization in vvRKHS

Tests to solve the pure translation problem with the Gauss kernel and different start values σ



(a) $\sigma = 10$ (b) $\sigma = 10$

$\Rightarrow$ Approximation quality also depends on the diffusion coefficients

Summary and Outlook

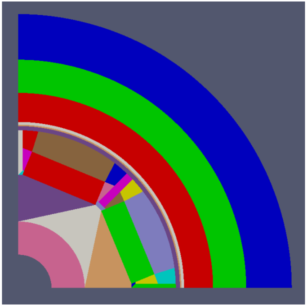
Summary:

- Short recap
- Numerical results with more complex domains
- Numerical results with heuristic approach
- Numerical tests in vvRKHS

Summary and Outlook

Next steps:

- Applying the optimization procedure to electrical machines



(a) Quarter of a cross section of an electric motor in Paraview

$$\min_{\Omega} J(u) := \int_{\Gamma} |B(u) \cdot n_g - B_d|^2 ds$$
$$= \int_{\Gamma} |\nabla u \cdot \tau_g - B_d|^2 ds$$

subject to

$$a(u, v) = \langle F, v \rangle \qquad \forall v \in H_0^1(D)$$

with

$$a(u, v) = \int_D \nu(x, |\nabla u|) \nabla u \cdot \nabla v \, dx,$$
$$\langle F, v \rangle = \int_{\Omega_M} M^{\perp} \cdot \nabla v \, dx$$