T U T O R I A L

"Computational Mechanics"

to the lecture

"Numerical Methods in Continuum Mechanics 1"

Tutorial 03-04

Date: Thursday, 21 April 2016
Time : $10^{\underline{15}}$ – $11^{\underline{45}}$
Room : S2 354

1.4 Theorem of Babuska and Aziz

Theorem 1.5: (Babuška und Aziz, 1972)
Let U and V be Hilbert spaces. Then the linear map (operator) $A : U \longmapsto V^*$ is an isomorphism (bijective, A and A^{-1} continuous) if and only if (= iff) the corresponding bilinear form $a(.,.) : U \times V \to \mathbb{R}$ fulfills the following conditions:

1. continuity, i.e. $\exists\, \mu_2 = \text{const.} > 0$:
$$(8) \quad |a(u,v)| \leq \mu_2 \|u\|_U \|v\|_V \;\forall u \in U, \forall v \in V,$$
2. inf-sup-condition, i.e. $\exists\, \mu_1 = \text{const.} > 0$:
$$(9) \quad \inf_{u\in U} \sup_{v \in V} \frac{a(u,v)}{\|u\|_U \|v\|_V} \geq \mu_1,$$
3. $\forall v \in V \setminus \{0\}\; \exists u \in U$:
$$(10) \quad \text{a(u,v)} \neq 0.$$

10* Let us assume that the sufficient conditions (8) - (10) of the Babuška-Aziz-Theorem 1.5 are fulfilled. Let us consider the variational problem: find $u \in U$ such that
$$A(u,v) = < F, v >, \quad \forall v \in V, \tag{1.10}$$
where the bilinear form $A(u,v)$ and the linear form $< F, v >$ are defined by the identities
$$A(u,v) = < A^* J A u, v >, \quad \forall u, v \in U \tag{1.11}$$
$$< F, v > = < A^* J f, v >, \quad \forall v \in U, \tag{1.12}$$
respectively. Here $A^* : V \longrightarrow U^*$ denotes the adjoint to $A : U \longrightarrow V^*$ operator, and $J : V^* \longrightarrow V$ is the Riesz isomorphism between the Hilbert spaces V^* and V. Show that the linear form $< F, . >$ and bilinear form $A(.,.)$ fulfil the assumption of the Lax-Milgram-Theorem and provide the ellipticity and the boundedness constants of the bilinear form $A(.,.)$!

1.5 Nonlinear Variational Problems

11 Let us consider the abstract nonlinear variational problem (15) from Transparency 04 under the assumption made there. Show that there exists a unique solution $u \in V_0$ of the nonlinear variational problem (15) and that the fixed point iteration (17) converges to this solution !

12 Let us consider the abstract nonlinear variational problem (15) from Transparency 04 under the assumption made there, and its finite element approximation: find $u_h \in V_{0h} \subset V_0$ such that

$$a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_{0h}. \tag{1.13}$$

Show the Cea-like discretization error estimate

$$\|u - u_h\|_{V_0} \leq \frac{\mu_2}{\mu_1} \inf_{w_h \in V_{0h}} \|u - w_h\|_{V_0}, \tag{1.14}$$

where the μ_1 and μ_2 are the monotonicity and the Lipschitz constants, respectively.

2 Analysis and Numerics of Mixed Variational Problems

2.1 Mixed Variational Problems

Consider the mixed variational problem: Find $u \in X$ and $\lambda \in \Lambda$, such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \langle f, v\rangle \,, \quad \forall v \in X \,, \\ b(u,\mu) &= \langle g, \mu \rangle \,, \quad \forall \mu \in \Lambda \,. \end{aligned}$$

In order to guarantee a unique existence of the solution (see Theorem 2.4 (*Brezzi*) in the lectures) one has to verify the following conditions:

1. The linear forms f and g are continuous, i.e.,

$$f \in X^* \,, \quad g \in \Lambda^* \,, \tag{2.15}$$

2. the bilinear forms $a(\cdot,\cdot) : X \times X \to \mathbf{R}$ and $b(\cdot,\cdot) : X \times \Lambda \to \mathbf{R}$ are continuous, i.e., $\exists$ positive constants α_2, β_2:

$$|a(u,v)| \leq \alpha_2 \|u\|_X \|v\|_X \,, \quad \forall u, v \in X \,, \tag{2.16}$$

$$|b(v,\mu)| \leq \beta_2 \|v\|_X \|\mu\|_\Lambda \,, \quad \forall v \in X \,, \forall \mu \in \Lambda \,, \tag{2.17}$$

3. LBB (Ladyshenskaja – Babuska – Brezzi) condition: $\exists$ positive constant β_1:

$$\inf_{\substack{\mu \in \Lambda \\ \mu \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_\Lambda} \geq \beta_1 \,, \tag{2.18}$$

4. Ker B–ellipticity, i.e., $\exists$ positive constant α_1:

$$a(v,v) \geq \alpha_1 \|v\|_X^2 \,, \quad \forall v \in \operatorname{Ker} B \,, \tag{2.19}$$

where $\operatorname{Ker} B = \{v \in X \mid Bv = 0 \text{ (in } \Lambda^*)\} = \{v \in X \mid \underbrace{b(v,\mu)}_{=\langle Bv\,,\,\mu\rangle} = 0 \,, \forall \mu \in \Lambda\}$.

13 Consider the mixed formulation of the 1st BVP of the biharmonic equation (see Example 1.3 in the lectures, and Exercise 9 of the tutorials):
Find $w \in X := H^1(\Omega)$ and $u \in \Lambda := H_0^1(\Omega)$ such that there holds

$$\begin{aligned} \int_\Omega w\, m \, \mathrm{d}x - \int_\Omega \nabla m \cdot \nabla u \, \mathrm{d}x &= 0\,, \quad \forall m \in X\,, \\ -\int_\Omega \nabla w \cdot \nabla v \, \mathrm{d}x \qquad\qquad &= \int_\Omega f\, v \, \mathrm{d}x\,, \quad \forall v \in \Lambda\,, \end{aligned}$$

Solution 21 Show that for this problem, the conditions (2.16) and (2.18) are satisfied ! What can you say about (2.19) ?

14 Consider the Stokes problem (see Example 1.1 in the lectures): Find $u \in X := [H_0^1(\Omega)]^3$ and $p \in \Lambda := \{q \in L_2(\Omega) \mid \int_\Omega q \, \mathrm{d}x = 0\}$ such that there holds

$$\begin{aligned} \frac{1}{\mathrm{Re}} \int_\Omega \nabla u : \nabla v \, \mathrm{d}x - \int_\Omega \operatorname{div} v \, p \, \mathrm{d}x &= \int_\Omega f\, v \, \mathrm{d}x\,, \quad \forall v \in X\,, \\ -\int_\Omega \operatorname{div} u \, q \, \mathrm{d}x \qquad\qquad &= 0\,, \quad \forall q \in \Lambda\,, \end{aligned}$$

where the Reynolds number Re is positive, and where : denotes the inner product $A : B = \sum_{i,j=1}^3 a_{ij}\, b_{ij}$, defined for matrices $A = (a_{ij})_{i,j=1,2,3}$ and $B = (b_{ij})_{i,j=1,2,3}$. Show that for this problem the conditions (2.16) – (2.19), except for the too difficult part (2.18), are satisfied.

15* Let X and Λ be real Hilbert spaces and $B : X \to \Lambda^*$ a bounded linear operator. Show that B satisfies the LBB-condition

$$\exists \beta_1 > 0 : \inf_{\substack{v \in \Lambda \\ v \neq 0}} \sup_{\substack{\tau \in X \\ \tau \neq 0}} \frac{\langle B\tau\,,\, v\rangle}{\|\tau\|_X \|v\|_\Lambda} \geq \beta_1\,,$$

if and only if there exists a positive constant c such that for all $v^* \in \Lambda^*$ there exists a $\tau \in X$ such that $B\tau = v^*$ and $\|\tau\|_X \leq c\,\|v^*\|_{\Lambda^*}$.