2.1.2. Trace Operators and Trace Spaces

see CEM04

- **Recall:** $H^{1/2}(\Gamma) := tr_\Gamma H^1(\Omega)$

Ω is $\mp \wedge$ Lip

- The trace space of $H^1(\Omega) = \overline{C^m(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$ is naturally defined by

$H^{1/2}(\Gamma) = tr_\Gamma H^1(\Omega)$

$$H^{1/2}(\Gamma) := \{ tr_\Gamma u \in L_2(\Gamma) : u \in H^1(\Omega) \} \subset L_2(\Omega)$$

equipped with the <u>norm</u> $(d=3)$

(1) $$\|w\|^2_{H^{1/2}(\Gamma)} := \|w\|^2_{L_2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|w(x)-w(y)|^2}{|x-y|^2} \, dx\, dy = (w,w)_{H^{1/2}(\Gamma)}$$

and the corresponding scalar product ($\rightarrow$ H-space), where the trace operator

$$tr_\Gamma u = \gamma_0 u = \gamma_D u = u|_\Gamma : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma)$$

is constructed as follows:

1) $(tr_\Gamma u)(x) := u(x) \; \forall x \in \Gamma \; \forall u \in H^1(\Omega) \cap C(\overline{\Omega})$.

2) Prove that $\exists \, c_T = const > 0$:

(2) $$\|tr_\Gamma u\|_{H^{1/2}(\Gamma)} \le c_T \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

3) Closure principle: Since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, $tr_\Gamma u$ is well defined for all $u \in H^1(\Omega)$ and (2) is valid for all $u \in H^1(\Omega)$.

We call (2) also <u>trace theorem</u>.

- The <u>inverse trace theorem</u> (extension theorem) is also valid: $\forall w \in H^{1/2}(\Gamma) \; \exists \, u \in H^1(\Omega)$:

$u = Ew$

(3) $$tr_\Gamma u = w \quad \text{and} \quad \|u\|_{H^1(\Omega)} \le c_E \|w\|_{H^{1/2}(\Gamma)}$$

with some <u>universal</u> positive constant $c_E = const > 0$.

- From (2) and (3), we immediately obtain

(4) $$\frac{1}{C_T}\|w\|_{H^{1/2}(\Gamma)} \overset{(2)}{\leq} \inf_{\substack{u\in H^1(\Omega)\\ tr_\Gamma u = w}} \|u\|_{H^1(\Omega)} \overset{(3)}{\leq} c_E \|w\|_{H^{1/2}(\Gamma)} \quad \forall w\in H^{1/2}(\Gamma)$$

i.e. the inf-norm $|||w|||_{H^{1/2}(\Gamma)}$ is an equivalent to $\|\cdot\|_{H^{1/2}}$ norm (norms).

- Dual space: $H^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))^*$

Duality product: $\langle\cdot,\cdot\rangle_{H^{-1/2}\times H^{1/2}} : H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)\to\mathbb{R}$

Remark: $\langle \partial_s, v\rangle = \int_\Gamma \partial_s v\, ds \leq \|\partial_s\|_{H^{-1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)} \leq \|\partial_s\|_{H^{-1/2}} c_T \|v\|_{H^1(\Omega)} \quad \forall v\in H^1(\Omega)$

(IbP) if $\partial_s \in L_2(\Gamma) \subset H^{-1/2}(\Gamma)$!

- Integration by parts: $\forall u\in H^1(\Omega)$, $\forall \varphi\in (C^\infty(\bar\Omega))^3$:

(5) $$\int_\Omega \nabla u\cdot\varphi\, dx + \int_\Omega u\cdot \operatorname{div}\varphi\, dx = \int_\Gamma tr_\Gamma u\cdot\varphi\cdot n\, ds$$

$\overset{\circ}{H}{}^1(\Omega) := \{v\in H^1(\Omega) : tr_\Gamma(u) = 0 \text{ on } \Gamma\} = \overline{\overset{\circ}{C}{}^\infty(\Omega)}^{\|\cdot\|_1}$.

$\overset{\circ}{H}{}^1(\Omega)$

DD

- **Lemma 2.3:** Let $\Omega_1,\dots,\Omega_m$ be a non-overlapping domain decomposition of Ω, i.e. $\bar\Omega = \cup\bar\Omega_i$, $\Omega_i\cap\Omega_j = \emptyset$, $i\neq j$.

Let $u_i\in H^1(\Omega_i)$, $i=\overline{1,m}$, such that

$$tr_{\Gamma_{ij}} u_i = tr_{\Gamma_{ij}} u_j \quad \forall \Gamma_{ij} = \partial\Omega_i\cap\partial\Omega_j : meas_{d-1}\Gamma_{ij} > 0.$$

Then the piecewise defined function

$$u := \{u|_{\Omega_i} = u_i,\ i=\overline{1,m}\} \in H^1(\Omega) \text{ and } (\nabla u)|_{\Omega_i} = \nabla u_i,\ i=\overline{1,m}.$$

Proof: Let $g_i = \nabla u_i$ be the local weak gradients.

Set $g := g_i$ on Ω_i $\forall i=\overline{1,m}$, i.e. $g\in L_2(\Omega)$. Then, $\forall\varphi\in(\overset{\circ}{C}{}^\infty(\Omega))^3$, we have

$$-\int_\Omega u \operatorname{div}\varphi\, dx = -\sum_{i=1}^m \int_{\Omega_i} u_i \operatorname{div}\varphi\, dx \overset{(5)}{=}$$

$$= \sum_{i=1}^m \Big[\int_{\Omega_i} \nabla u_i\cdot\varphi\, dx - \int_{\partial\Omega_i} tr_{\partial\Omega_i} u_i\cdot\varphi\cdot n_i\, ds\Big]$$

$$= \sum \int_{\Omega_i} g_i\cdot\varphi\, dx - \sum_{\Gamma_{ij}}' \int_{\Gamma_{ij}} \underbrace{(tr_{\Gamma_{ij}} u_i - tr_{\Gamma_{ij}} u_j)}_{=0}\varphi\cdot n_i\, ds$$

$$= \sum' \int_{\Omega_i} g_i\cdot\varphi\, dx = \int_\Omega g\cdot\varphi\, dx,$$

i.e. g is the weak gradient of u in Ω (Def. 2.1!). $\square$

Traces of functions from H(div):

- Starting point ≡ div:IbyP-formula:

$$\int_\Omega \operatorname{div} q \cdot v \, dx = -\int_\Omega q \bullet \nabla v \, dx + \int_\Gamma q \bullet n \cdot v \, ds \text{, i.e.}$$

$$\int_\Gamma \underbrace{q\bullet n}_{=: tr_n q \,\in H^{-1/2}(\Gamma)} \cdot \underbrace{v}_{=: tr_\Gamma v \,\in H^{1/2}(\Gamma)} ds := \int_\Omega (\operatorname{div} q \cdot v + q \bullet \nabla v)\, dx$$

$\langle \quad , \quad \rangle$ $q \in H(\operatorname{div})$ and $v \in H^1(\Omega)$

$$\langle tr_n q, tr_\Gamma v \rangle_{H^{-1/2} \times H^{1/2}} := \int_\Omega (\operatorname{div} q \cdot v + q \bullet \nabla v)\, dx \quad \forall q \in H(\operatorname{div}) \, \forall v \in H^1$$

- **Theorem 2.4:** (H(div) trace theorem)

There exists a unique continuous operator

(6) (6) $tr_n : H(\operatorname{div}) \longrightarrow H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))^*$

such that

$\Omega \uparrow \wedge$ Lip

$$tr_n q(x) = q(x) \bullet n(x) \quad \forall x \in \Gamma \text{ (a.e.)}$$

for all functions $q \in H(\operatorname{div}) \cap [C^1(\overline{\Omega})]^3$.

Proof: The construction steps 1) – 3) of the H^1-case.
We have to prove continuity on a smooth, dense subspace. Let $q \in H(\operatorname{div}) \cap [C^1(\overline{\Omega})]^3$.
Then we have

(7) (7) $$\|tr_n q\|_{H^{-1/2}(\Gamma)} = \sup_{w \in H^{1/2}} \frac{\overbrace{\int_\Gamma q\bullet n \cdot w \, ds}^{\langle q\cdot n, w\rangle}}{\|w\|_{H^{1/2}}} \qquad tr\, v = w \quad (\geq C_{E1}^{-1} \|v\|_{H^1(\Omega)})$$

H^1 inv. tr. th. (3) — inverse trace theorem = extension theorem

$$\leq C_{E1} \sup_{v \in H^1(\Omega)} \frac{\int_\Gamma q\bullet n \cdot tr_\Gamma v \, ds}{\|v\|_{H^1(\Omega)}} =$$

$$= \overset{C_T}{\overset{\shortparallel}{C_{E1}}} \sup_{v \in H^1(\Omega)} \frac{\int_\Omega (\operatorname{div} q \cdot v + q \bullet \nabla v)\, dx}{\|v\|_{H^1(\Omega)}} \leq \overset{C_T(H(\operatorname{div})) = C_E(H^1)}{\overset{\shortparallel}{C_{E1}}} \|q\|_{H(\operatorname{div})}.$$

Closure principle. **q.e.d.**

extension

- **Theorem 2.5:** (H(div) inverse trace theorem = ext. Th.)

Let $q_n \in H^{-1/2}(\Gamma)$. Then there exists an $q \in H(\mathrm{div})$ such that

$$(8) \quad tr_n q = q_n \text{ and } \|q\|_{H(\mathrm{div})} \le c_E \|q_n\|_{H^{-1/2}(\Gamma)}.$$

$\int_\Omega \mathrm{div}\, q \cdot 1\, dx = 0 + \int_\Gamma q_n \cdot 1\, ds$

If q_n satisfies $\langle q_n, 1\rangle = 0$, then there exists an extension $q \in H(\mathrm{div})$ such that $\mathrm{div}\, q = 0$.

Proof: Consider the weak solution of the Neumann problem

$$(9) \quad \begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = q_n & \text{on } \Gamma \end{cases}$$

Since $q_n \in H^{-1/2}(\Gamma)$, we get from Lax-Milgram (...) that $\exists!\, u \in H^1(\Omega)$: (mms)

$$(10) \quad \|\nabla u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} \le c \|q_n\|^2_{H^{-1/2}(\Gamma)}.$$

Now, set $q = \nabla u$. We observe that

(weak) $\mathrm{div}\, q = \mathrm{div}\, \nabla u = \Delta u \overset{(9)}{=} u \in L_2(\Omega)$ (mms)

and

$$\|q\|^2_{L_2(\Omega)} + \|\mathrm{div}\, q\|^2_{L_2(\Omega)} = \|\nabla u\|^2_{L_2} + \|u\|^2_{L_2} \overset{(10)}{\le} c \|q_n\|^2_{H^{-1/2}(\Gamma)}$$

If $q_n : \langle q_n, 1\rangle = 0$, then consider

$$(11) \quad \begin{cases} -\Delta u = 0 \\ \frac{\partial u}{\partial n} = q_n \perp 1 \end{cases} \qquad \exists! \ (L\&M)$$

$\Rightarrow q = \nabla u \Rightarrow \mathrm{div}\, q = \mathrm{div}\, \nabla u = \Delta u \overset{(11)}{=} 0.$ **q.e.d.**

V12: 29 April 2010

Tut 04:

Tut 03

- **Exercise 2.6:** (cf. Lemma 2.3)

Let $\Omega_1, \dots, \Omega_m$ be a DD of $\overline{\Omega} = \cup \overline{\Omega}_i$: $\Omega_i \cap \Omega_j = \emptyset \ \forall i \ne j$.

Let $q_i \in H(\mathrm{div}, \Omega_i)$: $tr_{n_i,\Gamma_{ij}} q_i = tr_{n_i,\Gamma_{ij}} q_j \quad \forall \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ (↑)

Then $q := \{ q = q_i \text{ on } \Omega_i,\ i = \overline{1,m} \} \in H(\mathrm{div})$ and $(\mathrm{div}\, q)|_{\Omega_i} = \mathrm{div}\, q_i$

→ Tutorial 03, Ex. 12 or mms!

Traces of functions from H(curl):

- Starting point = curl IbyP-formula: → Tut 03, Ex. 11

$$\int_\Omega \operatorname{curl} u \cdot v \, dx = \int_\Omega u \cdot \operatorname{curl} v \, dx - \int_\Gamma (u \times n) \cdot v \, ds, \text{ i.e.}$$

$$= \int_\Omega u \cdot \operatorname{curl} v \, dx + \int_\Gamma u \cdot (v \times n) \, ds$$

$$\int_\Gamma \underbrace{u \times n}_{!!} \cdot \underbrace{v}_{!!} \, ds := \int_\Omega (u \cdot \operatorname{curl} v - \operatorname{curl} u \cdot v) \, dx$$

$$\langle \underset{\substack{\cap \\ H^{-1/2}(\Gamma)^3}}{tr_t u} ; \underset{\substack{\cap \\ H^{1/2}(\Gamma)^3}}{tr v} \rangle := \int_\Omega (u \cdot \operatorname{curl} v - \operatorname{curl} u \cdot v) \, dx \quad \forall u \in H(\operatorname{curl} u) \quad \forall v \in H^1(\Omega) \, ?$$

- **Theorem 2.6:** (H(curl) trace theorem)

There exist a unique continuous operator

(12) $\quad tr_t = tr_{t,\Gamma} : H(\operatorname{curl}) \longrightarrow [H^{-1/2}(\Gamma)]^3$

such that

$$tr_t u(x) = u(x) \times n(x) \quad \forall x \in \Gamma = \partial\Omega \ (a.e) \qquad \Gamma \text{ Lip}$$

for all function $u \in H(\operatorname{curl}) \cap [C(\bar\Omega)]^3$.

Proof: $\triangleq$ 1)-3) for tr $H^1(\Omega)$ (↑)

Let $u \in H(\operatorname{curl}) \cap [C(\bar\Omega)]^3$. Then we have

(13) $$\|tr_t u\|_{H^{-1/2}(\Gamma)} = \sup_{w \in H^{1/2}(\Gamma)} \frac{\int_\Gamma u \times n \cdot w \, ds}{\|w\|_{H^{1/2}(\Gamma)}} =$$

H^1 inv. trace

$$\overset{(3)}{\leq} C_E \sup_{v \in H^1(\Omega)} \frac{\int_\Gamma u \times n \cdot tr v \, ds}{\|v\|_{H^1(\Omega)}} =$$

$$= C_E \sup_{v \in H^1(\Omega)} \frac{\int_\Omega (u \cdot \operatorname{curl} v - \operatorname{curl} u \cdot v) \, dx}{\|v\|_{H^1(\Omega)}}$$

$$\leq C_E \|u\|_{H(\operatorname{curl})} \sup_{v \in H^1(\Omega)} \frac{\|v\|_{H(\operatorname{curl})}}{\|v\|_{H^1(\Omega)}} \overset{*}{\leq} C \|u\|_{H(\operatorname{curl})}$$

Closure principle! q.e.d.

$$H_{\parallel}^{-1/2}(\mathrm{div}_t, \Gamma) = tr_t\, H(\mathrm{curl}, \Omega)$$

- **Remark 2.7:** $\mathrm{curl}\, v \longrightarrow \nabla v$?

The stated trace theorem for $H(\mathrm{curl})$ is not sharp!
Thus, there is no inverse trace theorem!
The right norm is (Buffa & Ciarlet, 2001)

$$\|tr_t u\|_{H_{\parallel}^{-1/2}(\mathrm{div}_t,\Gamma)} := \left(\|tr_t u\|^2_{H^{-1/2}(\Gamma)} + \|\mathrm{div}_t\, tr_t u\|^2_{H^{-1/2}(\Gamma)} \right)^{1/2}$$

(replace $\|tr_t u\|_{H^{-1/2}}$ by $\|tr_t u\|_{H_{\parallel}^{-1/2}(\mathrm{div}_t)}$!),

which leads to an inverse trace theorem!

- **Exercise 2.8:** (cf. Lemma 2.3)

Let $\Omega_1, \dots, \Omega_m$ be a non-overlapping DD of Ω, i.e.
$\overline{\Omega} = \bigcup \overline{\Omega_i}$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$.
Let $u_i \in H(\mathrm{curl}, \Omega_i)$, $i = \overline{1,m}$, such that

$$tr_{t_{ij}, \Gamma_{ij}} u_i = tr_{t_{ij}, \Gamma_{ij}} u_j \quad \forall\, \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j : \mathrm{meas}_{d-1} \Gamma_{ij} > 0.$$

Then the piecewise defined function

$$u := \{ u|_{\Omega_i} = u_i,\ i = \overline{1,m} \} \in H(\mathrm{curl}, \Omega) \text{ and}$$

$$(\mathrm{curl}\, u)|_{\Omega_i} = \mathrm{curl}\, u_i \quad \forall i = \overline{1,m}.$$

$\longrightarrow$ Tutorial 03, Ex. 13 or mms!

---

*) $v = \nabla w \in H(\mathrm{curl})$, $w \in H(\mathrm{grad})$

$$\langle tr_t u, \underbrace{tr_t \nabla w}_{= \nabla_t w = \nabla w \times n} \rangle := \int_\Omega (u \cdot \underbrace{\mathrm{curl}\, \nabla w}_{=0} - \mathrm{curl}\, u \cdot \overset{v}{\nabla w})\, dx$$

$$\parallel$$

$$\langle -\mathrm{div}_t u, tr_\Gamma w \rangle := -\int_\Omega \mathrm{curl}\, u \cdot \nabla w \, dx$$

$$H_{\parallel}^{-1/2}(\mathrm{div}_t, \Gamma) = tr_t\, H(\mathrm{curl}, \Omega)$$