

TUTORIAL

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 02

Tuesday, 28 March 2017, Time: 10¹⁵ – 11⁴⁵, Room: HT 177F.

1.2 The linear elasticity problem

07 Show that, for the BVP of the first type ($\Gamma_1 = \Gamma$) and for the mixed BVP ($\text{meas}_2(\Gamma_1) > 0$ and $\text{meas}_2(\Gamma_2) > 0$) of the linear elasticity. the following statements are true:

1. $a(., .)$ is symmetric, i.e., $a(u, v) = a(v, u) \quad \forall u, v \in V$,
2. $a(., .)$ is nonnegative, i.e., $a(v, v) \geq 0 \quad \forall v \in V$,
3. $a(., .)$ is positive on $V_0 := \{v \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_1\}$, if $\text{meas}_2(\Gamma_1) > 0$, i.e., $a(v, v) > 0 \quad \forall v \in V_0 : v \neq 0$.

The equivalence of VF (9)_{VF} and MP (9)_{MP} then follows from the statements 1. and 2. above according to Section. 1.1. of the lecture.

08 Show that, for the BVP of the first type ($\Gamma_1 = \Gamma$) of 3D linear elasticity in the case of isotrop and homogeneous material, the assumptions of Lax-Milgram-Theorem are fulfilled. Provide constants μ_1 and μ_2 such that:

- 1) $F \in V_0^*$,
- 2a) $\exists \mu_1 = \text{const} > 0 : a(v, v) \geq \mu_1 \|v\|_{H^1(\Omega)}^2 \quad \forall v \in V_0$,
- 2b) $\exists \mu_2 = \text{const} > 0 : |a(u, v)| \leq \mu_2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall u, v \in V_0$.

○ Hint: to the proof of V_0 -ellipticity:

- $a(v, v) \geq 2\mu \int_{\Omega} \sum_{i,j=1}^3 (\varepsilon_{ij}(v))^2 dx$,
- Korn’s inequality for the BVP of the first type: $V_0 = [H_0^1(\Omega)]^3$, where $H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ auf } \Gamma\}$,
- FRIEDRICHS-inequality.

09 Formulate the iterative method (3) from Section 1.1 of the lecture for the first BVP of the linear elasticity in case of 3D homogeneous and isotrop material, i.e.,

$$u_{n+1} = u_n - \rho(JAu_n - JF) \text{ in } V_0 = (H_0^1(\Omega))^3, \quad (1.6)$$

for $n = 0, 1, 2, \dots$, and given $u_0 \in V_0$. Derive the weak form, i.e., the variational formulation, for the calculation of $u_{n+1} \in V_0$. Discuss two cases in which the scalar product in V_0 is defined as follows:

$$(u, v)_{V_0}^2 := \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V_0, \quad (1.7)$$

and

$$(u, v)_{V_0}^2 := \int_{\Omega} (\nabla u \cdot \nabla v + uv) \, dx \quad \forall u, v \in V_0. \quad (1.8)$$

10* Let us consider the variational formulation:

(*) Find $u \in V_g = V_0$: $a(u, v) = \langle F, v \rangle \quad \forall v \in V_0$ of a plane linear elasticity problem in $\Omega = (0, 1) \times (0, 1)$, where

$$\begin{aligned} V_0 = \{ & u = (u_1, u_2) \in V = [H^1(\Omega)]^2 : \\ & u_1 = 0 \text{ on } \Gamma_2 = \{0\} \times [0, 1] \cup \{1\} \times [0, 1], \\ & u_2 = 0 \text{ on } \Gamma_1 = [0, 1] \times \{0\} \cup [0, 1] \times \{1\}\}, \\ a(u, v) = & \int_{\Omega} D_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, dx = \int_{\Omega} \sigma_{kl}(u) \varepsilon_{kl}(v) \, dx, \\ \langle F, v \rangle = & \int_{\Omega} f_i v_i \, dx + \int_{\Gamma_1} ? \, ds + \int_{\Gamma_2} ? \, ds. \end{aligned}$$

Impose the right natural BCs! Give the classical formulation of (*).

1.3 Scalar elliptic problems of the fourth order

11 Show that the first biharmonic BVP

$$u \in V_0 := H_0^2(\Omega) : \int_{\Omega} \Delta u(x) \Delta v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx \quad \forall v \in V_0 \quad (1.9)$$

allows the application of Lax-Milgram-Theorem to show existence and uniqueness. Then formulate a minimization problem that is equivalent to the variational formulation above.